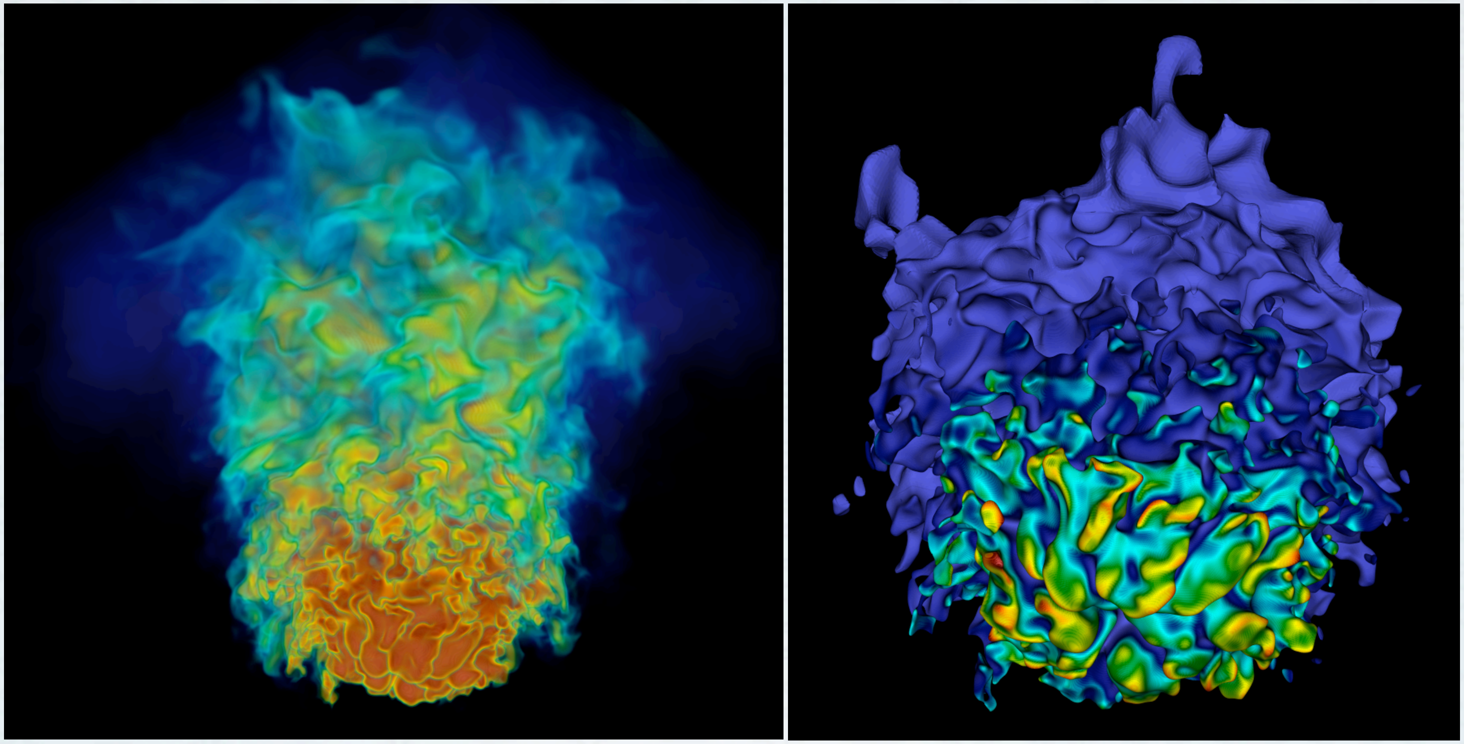
Scalar Topology in Visual Data Analysis

Theory and Motivational Applications

Isosurface Extraction and Scalar Field Visualization and Isosurfaces

- Scalar field: Assign scalar value (temperature, pressure etc.) to each location of domain
- Main visualization techniques: Direct volume rendering and isosurface extraction



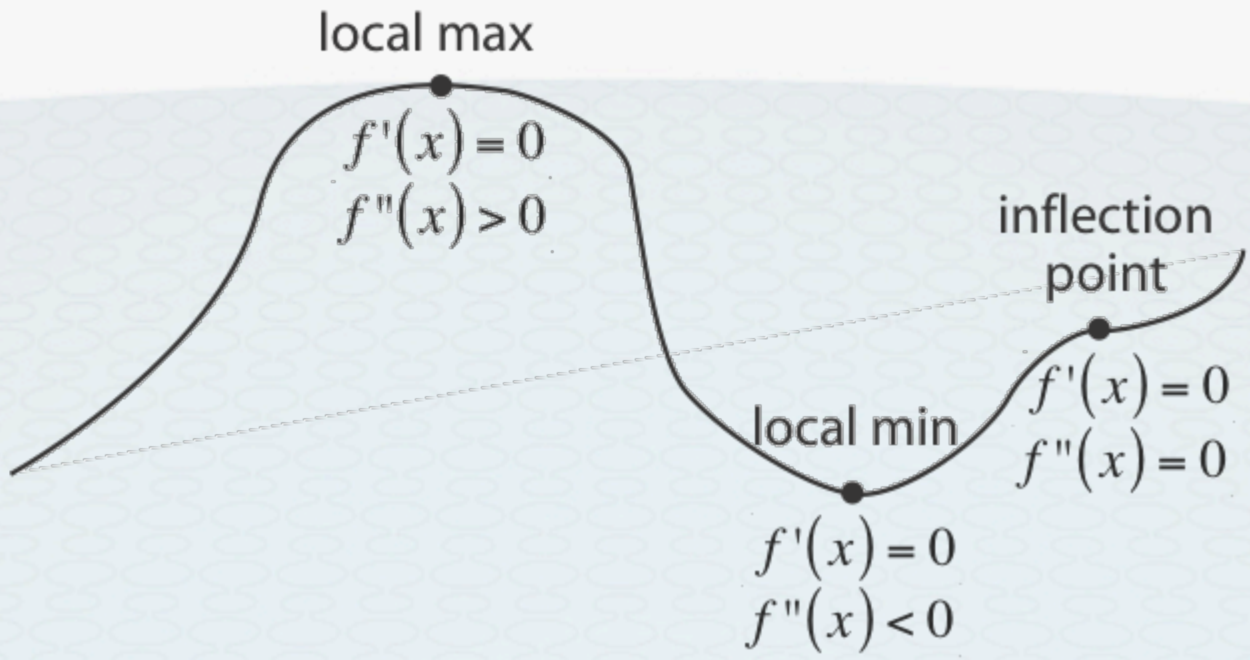
(Data courtesy of John Bell and Marc Day, LBNL CCSE)

Scalar Field Exploration with Isosurfaces

- Vary isovalue and observe isosurface changes
  - What type of "changes" can occur?
  - Which changes are relevant?
  - Can we determine where and when changes occur without extracting the actual isosurface?

- For 1D functions: Use differential calculus to identify maxima, minima, inflection points and sketch curve
- Equivalent considerations for isosurface extraction?

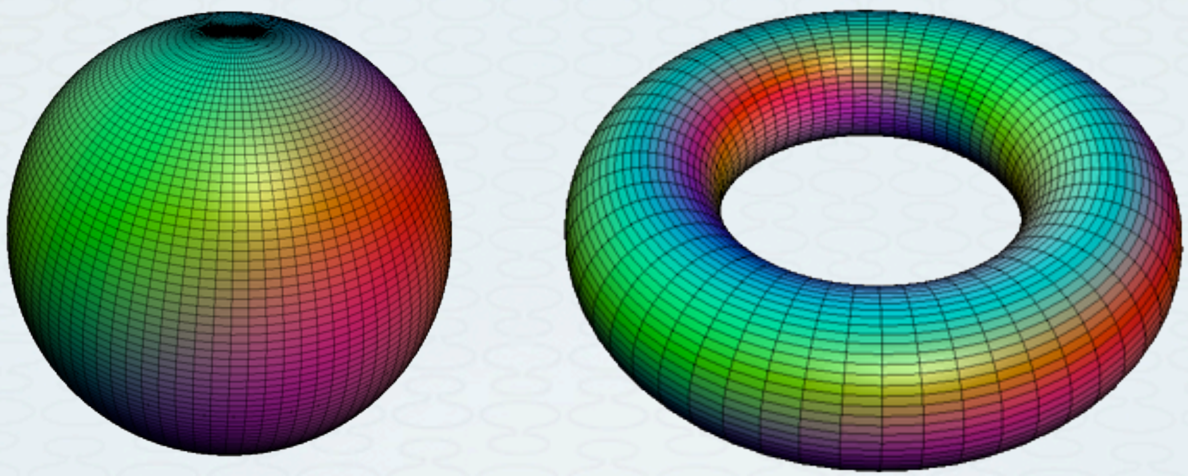
1D Refresher



- Collectively called *critical points*
- Partition function into *monotone* segments

Topology of Surfaces

- Properties that remain *invariant under elastic deformation*
- Topology of compact surface, e.g., defined by:
  - Number of connected components
  - Number of holes ➜ genus



Isosurface Topology Changes Occur at Critical Points

(Data courtesy of Hamish Carr, University College Dublin)

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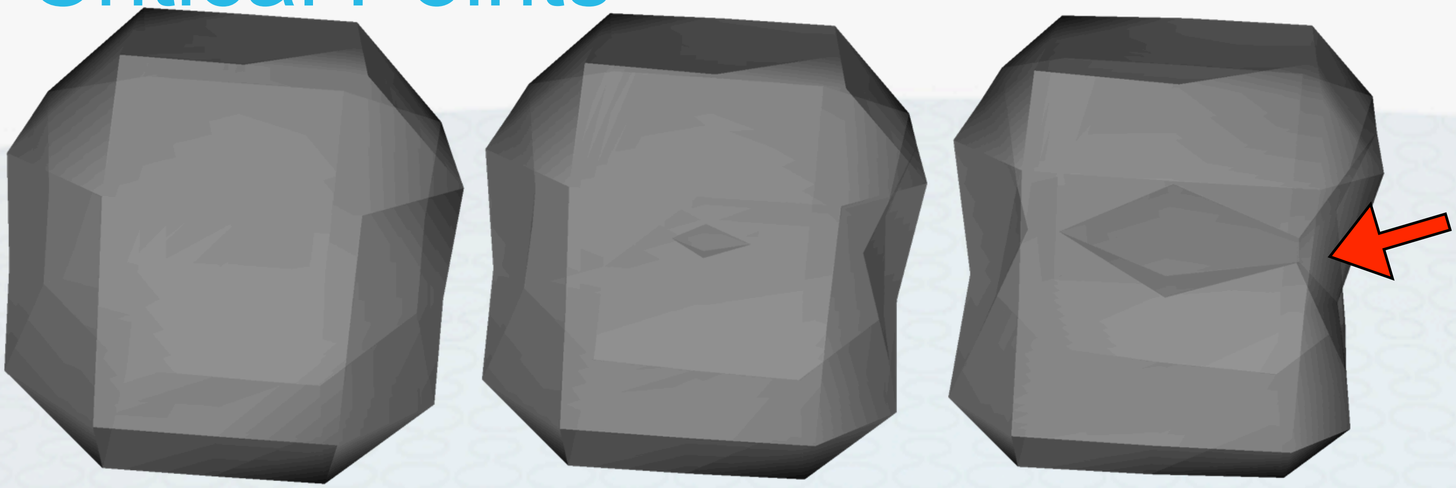

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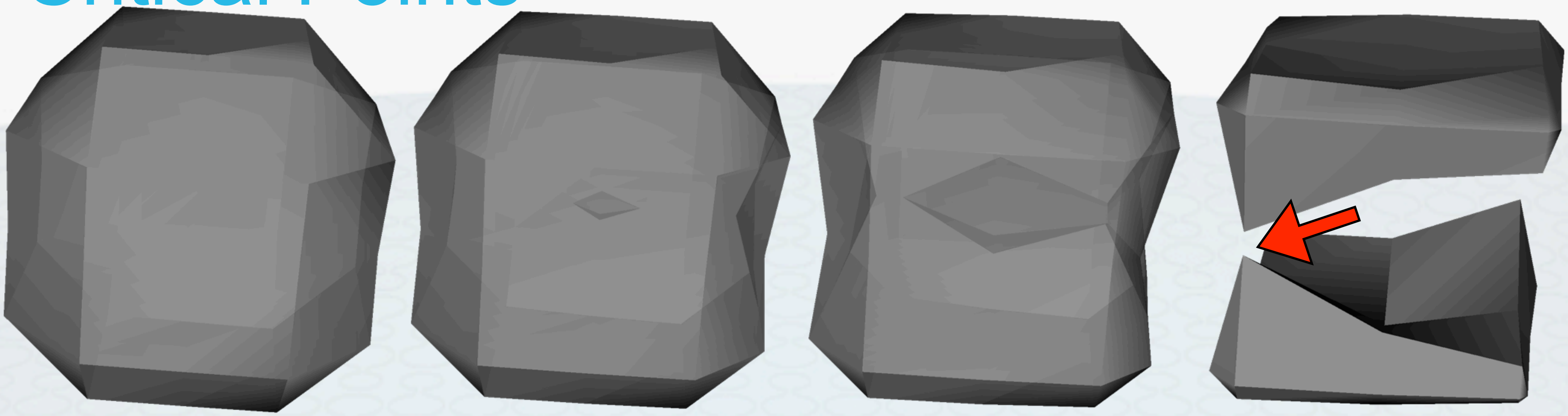

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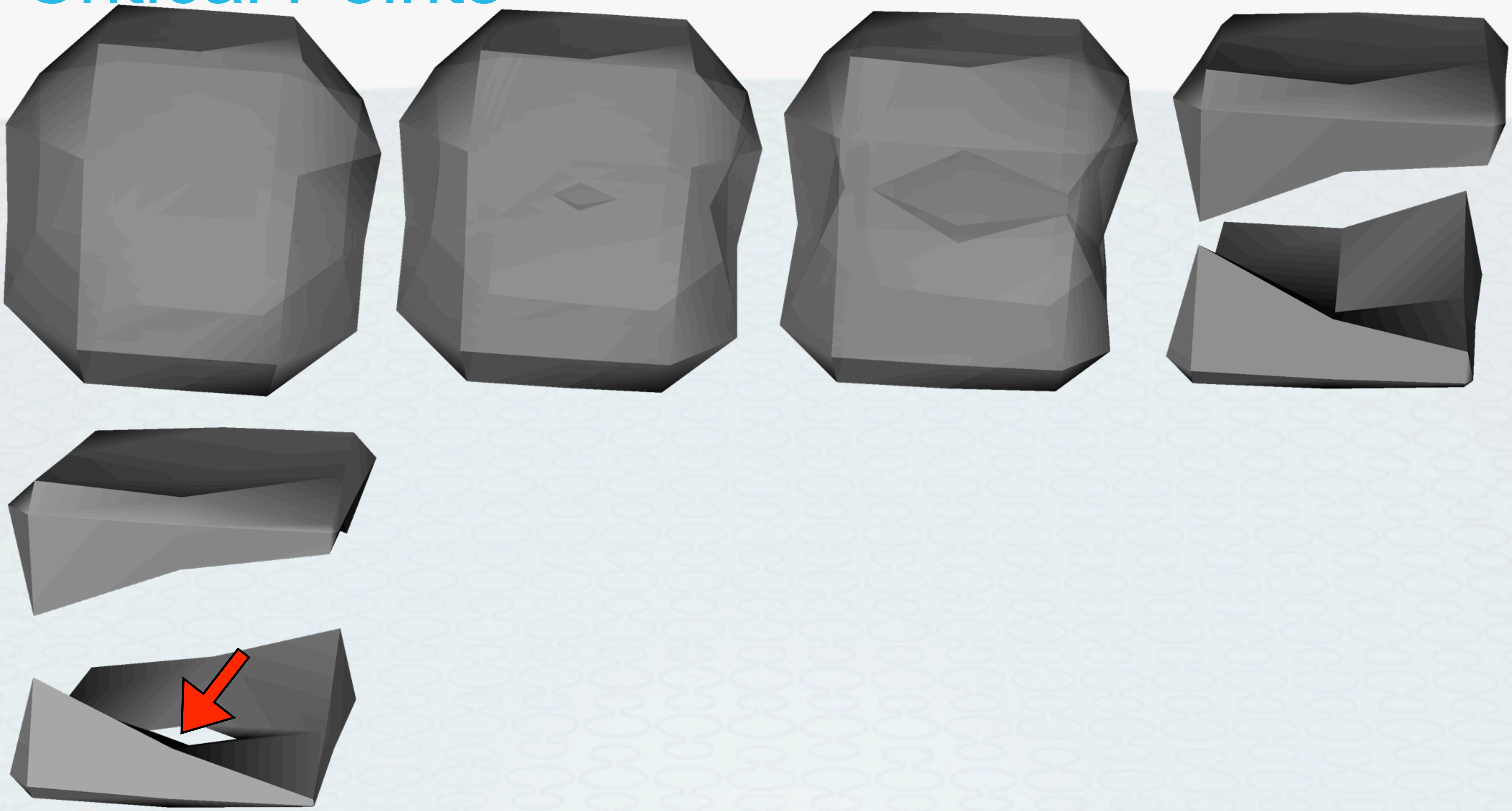

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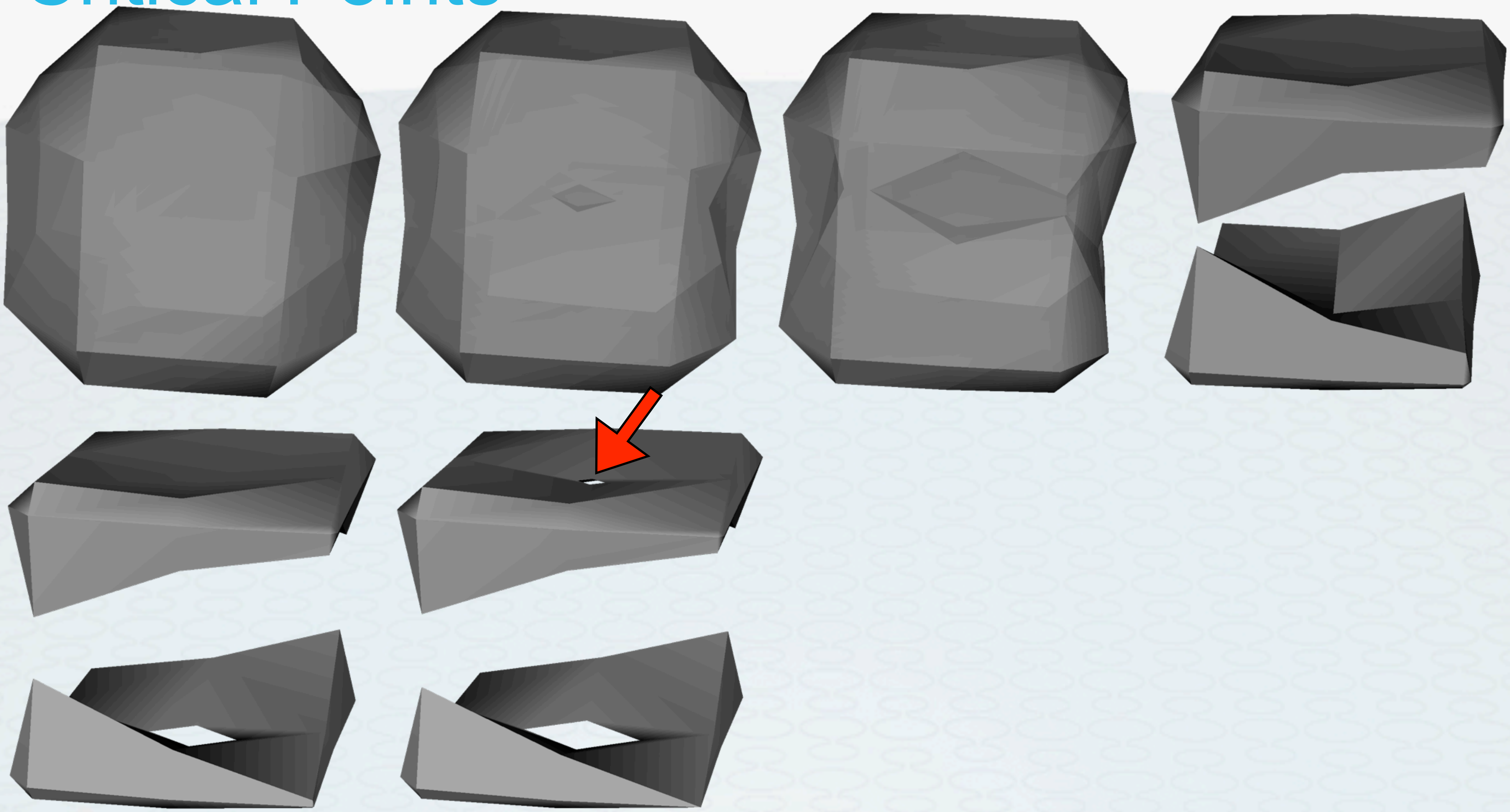

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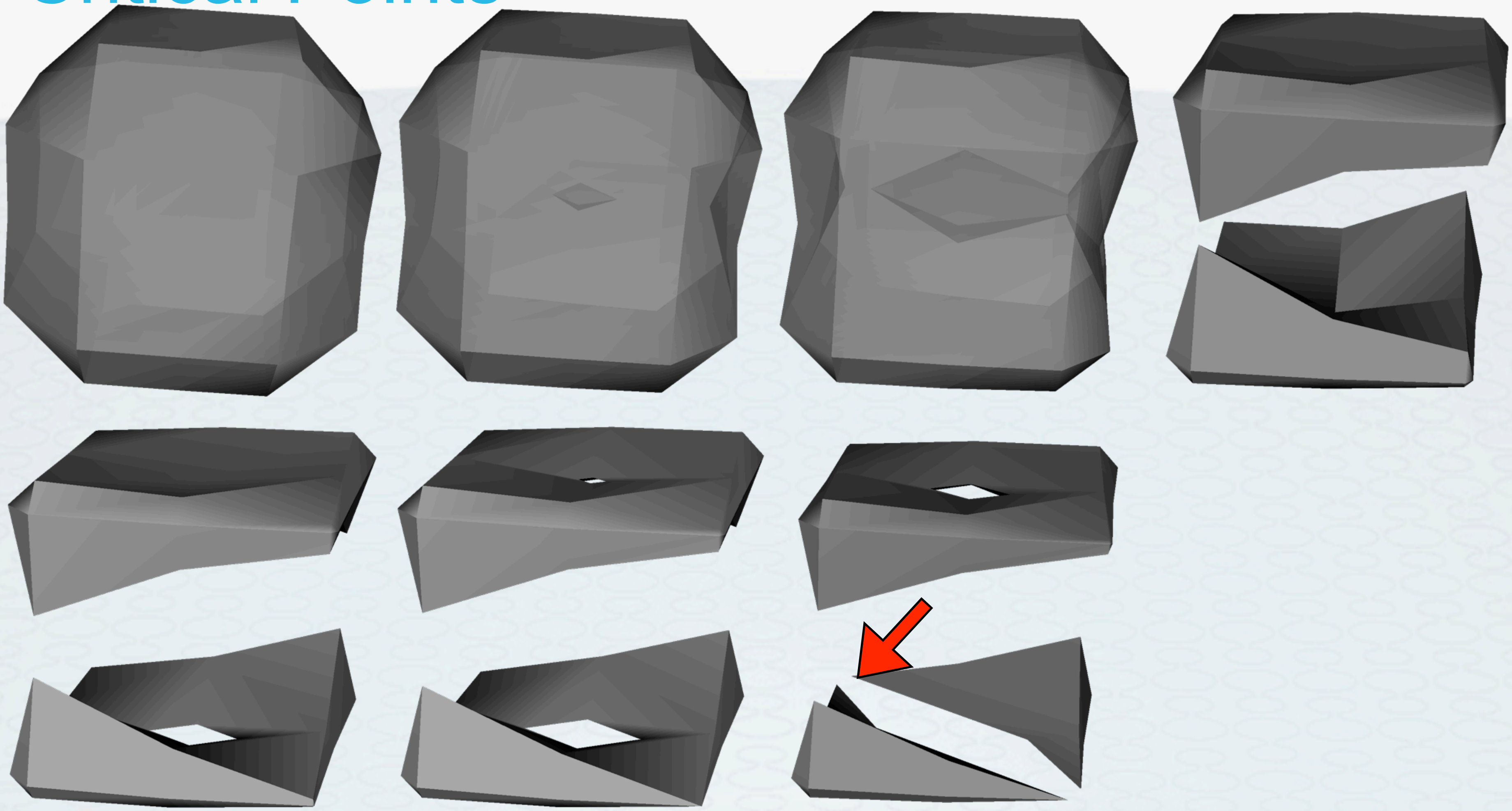

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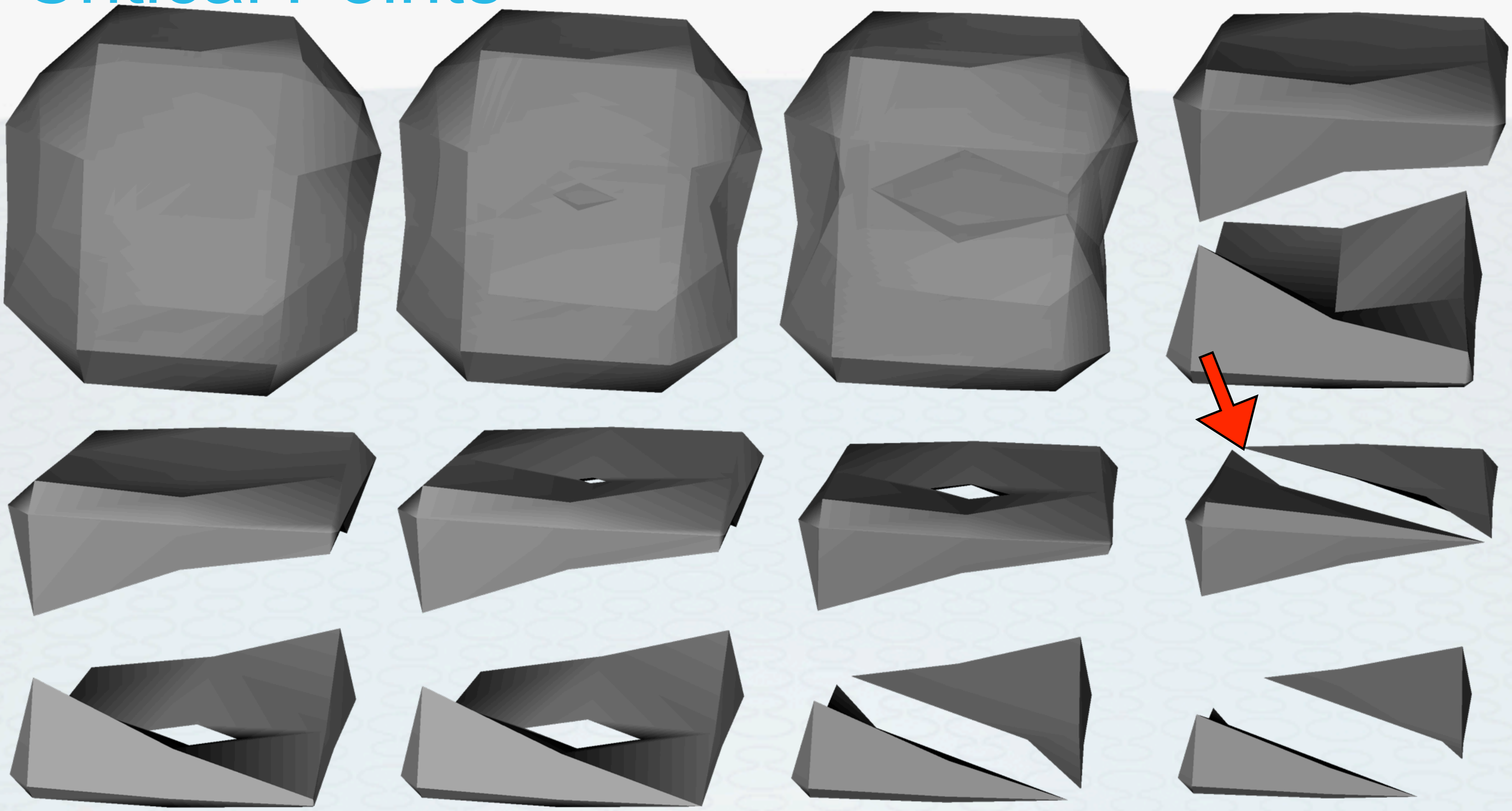

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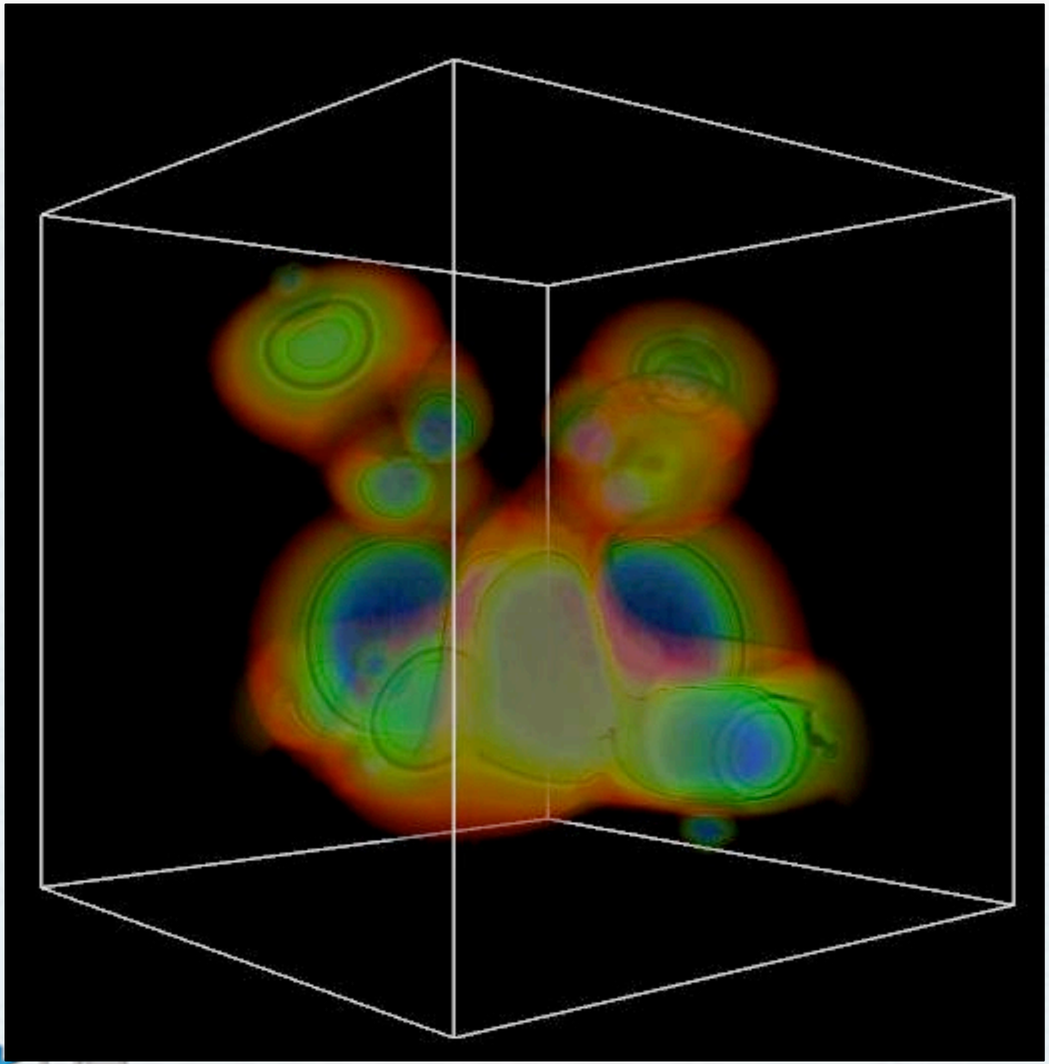
Topology-based Analysis Provides a Structural View of Scalar Functions



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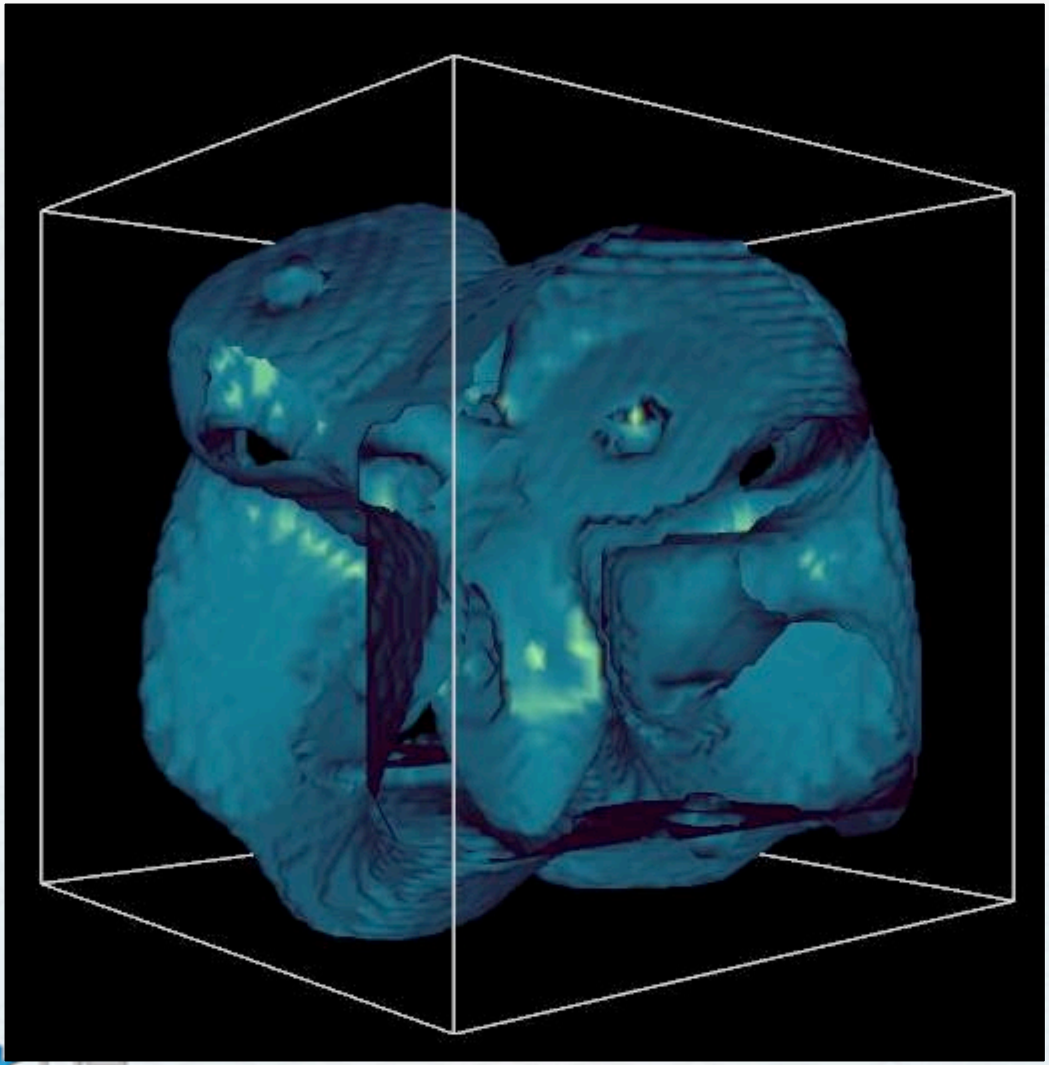

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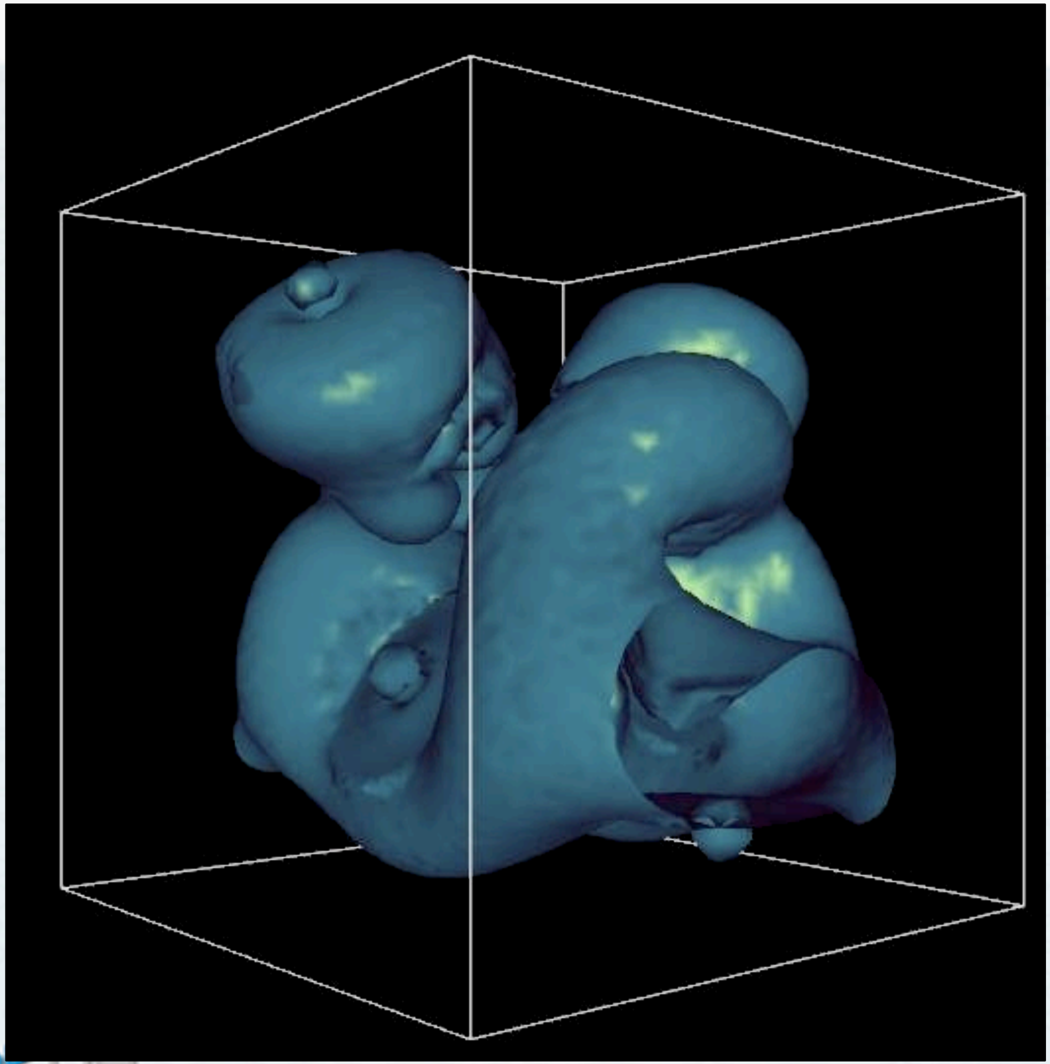

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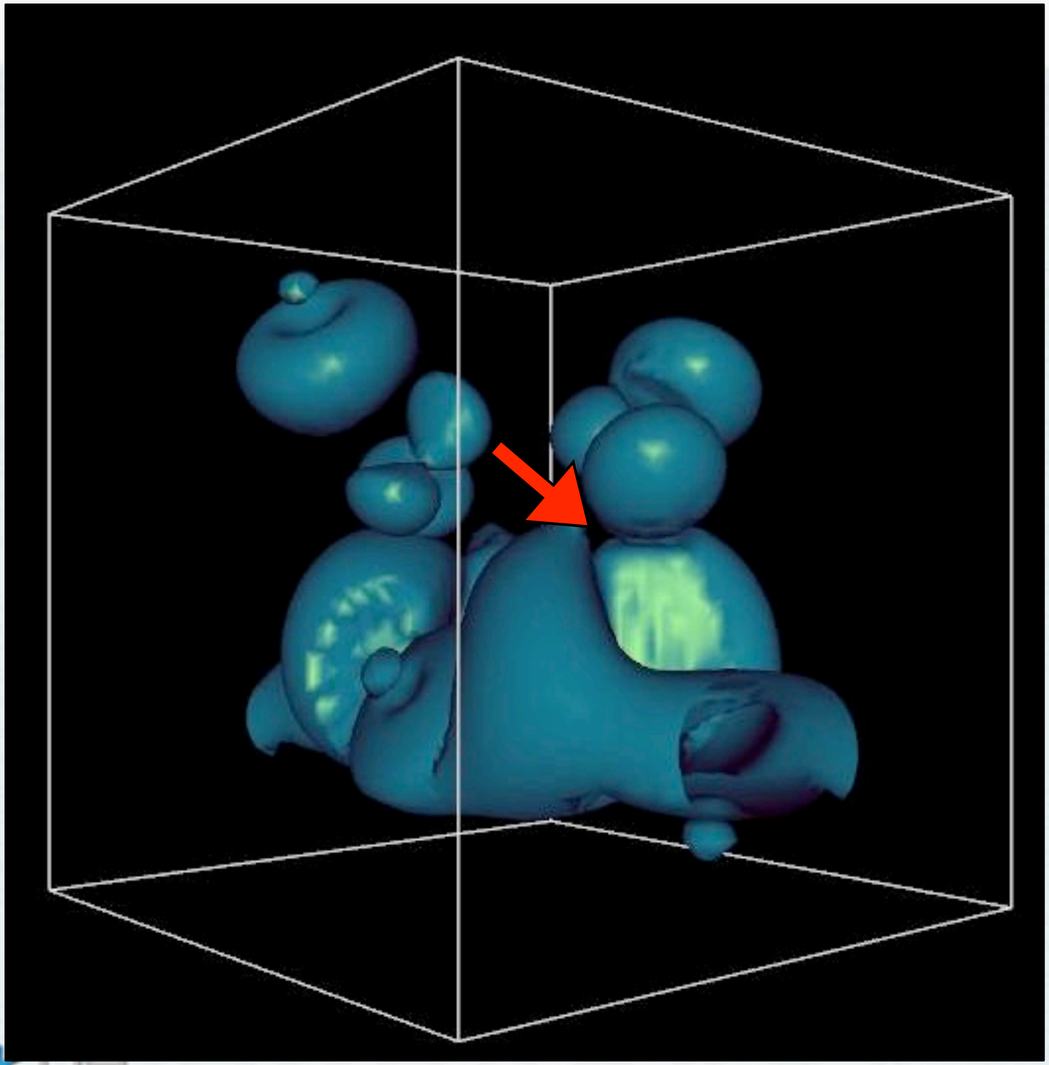

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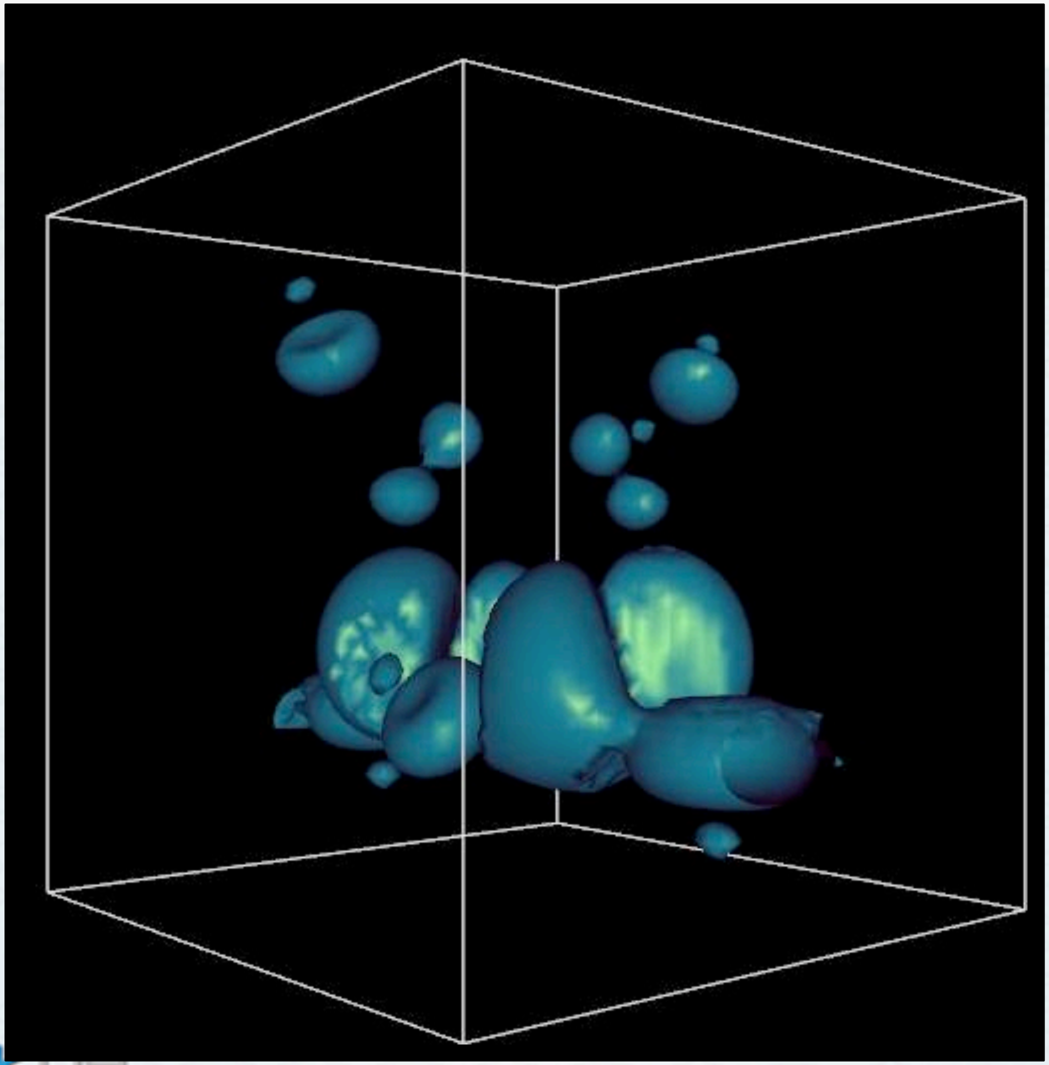

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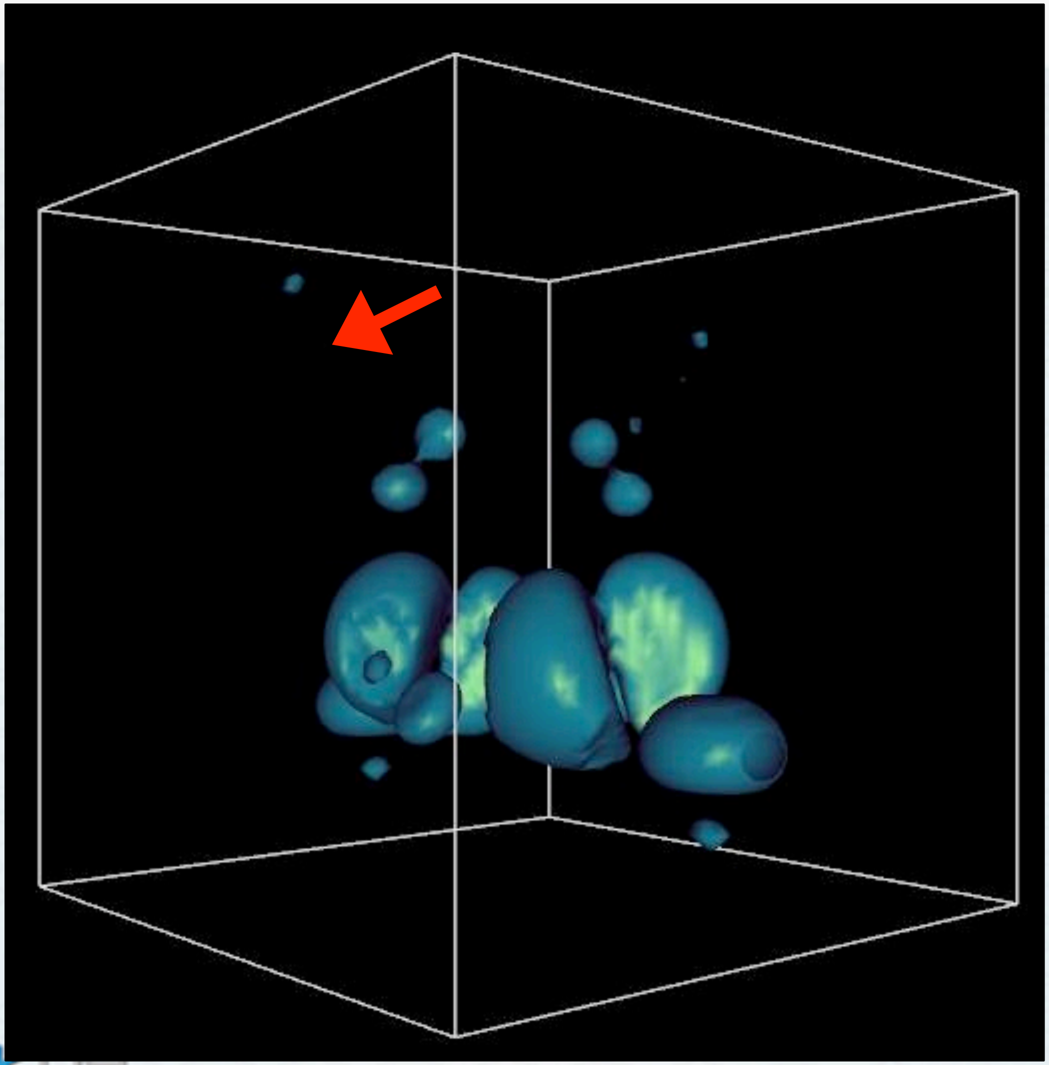

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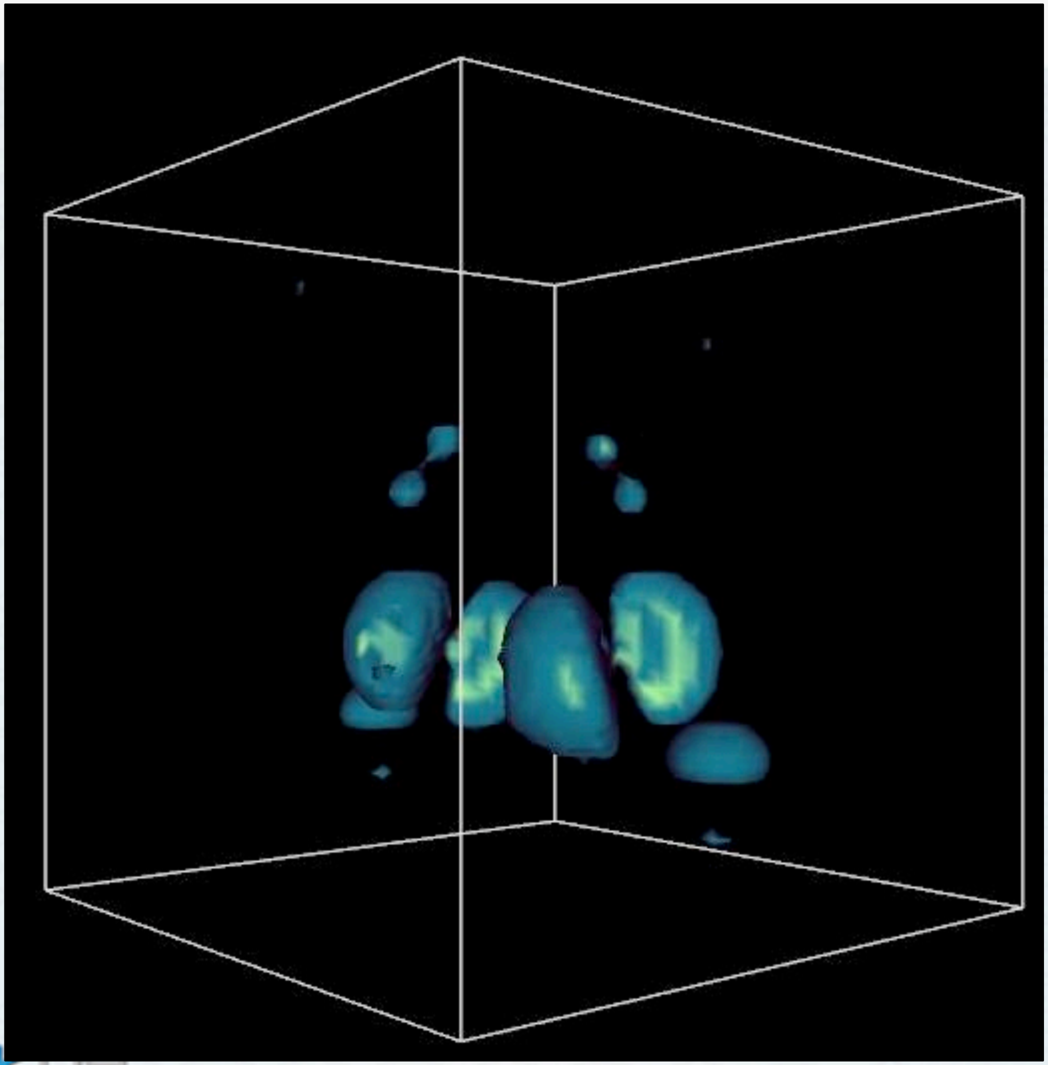
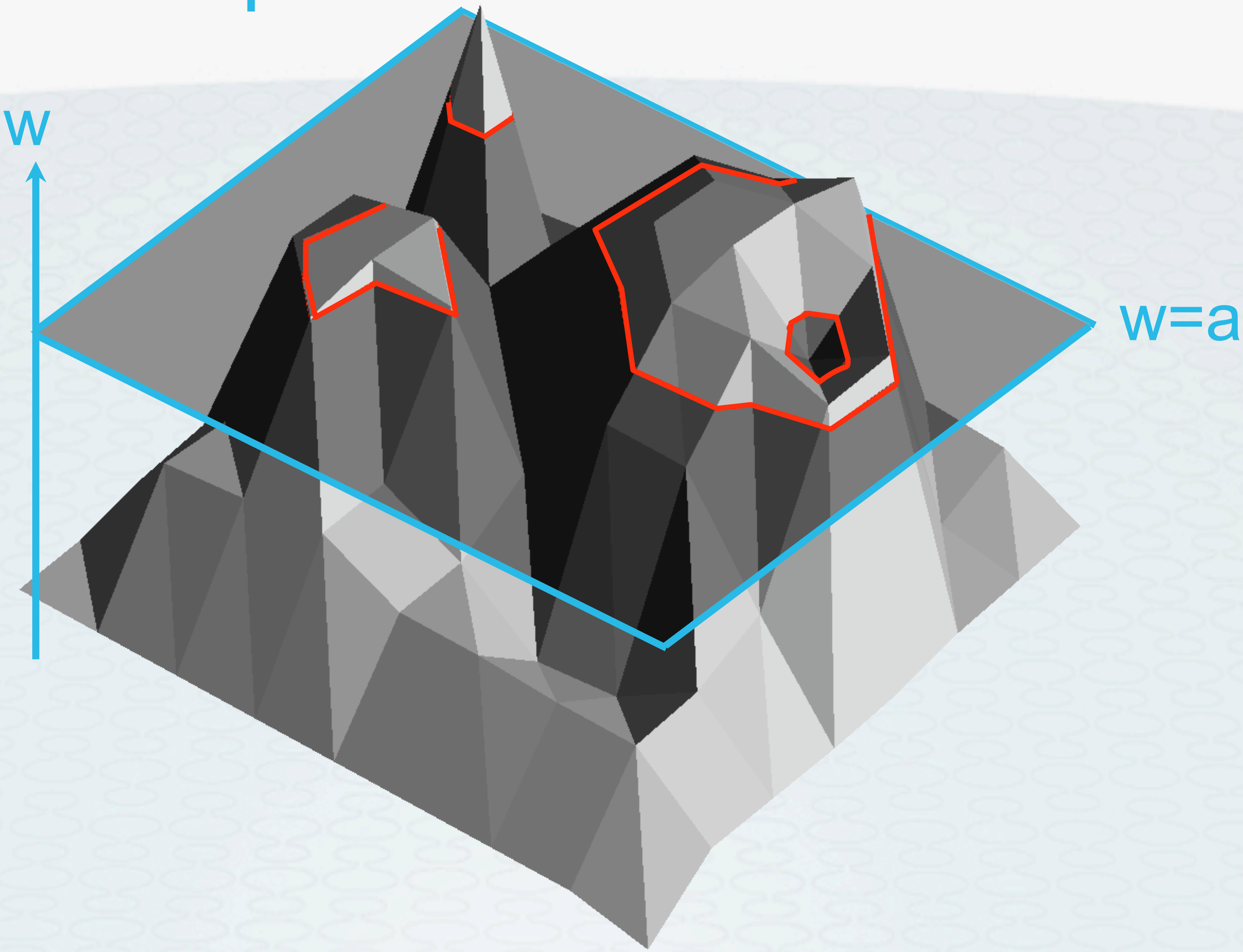


Criteria for Identifying Critical Points are based on Graph and "Isosurface"



Morse Theory Provides Analytical Criteria for Identifying Critical Points

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar valued function and $G(f) = \{p = (x, f(x)) = (p_x, p_y, p_z, p_w) \in \mathbb{R}^4\}$ its graph.

An isosurface $\{f(x) = a\}$ corresponds to an intersection $G(f) \cap \{w = a\}$.

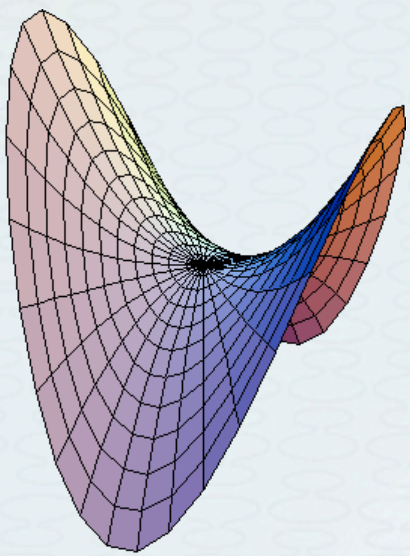
$x \in \mathbb{R}^3$ is a *critical point* if the tangential space to $G(f)$ in p is parallel to $\{w = p_w\}$, i.e., if the *gradient* $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ is zero.

Type of critical point is determined by the signs of the Eigenvalues of the Hessian

$$H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

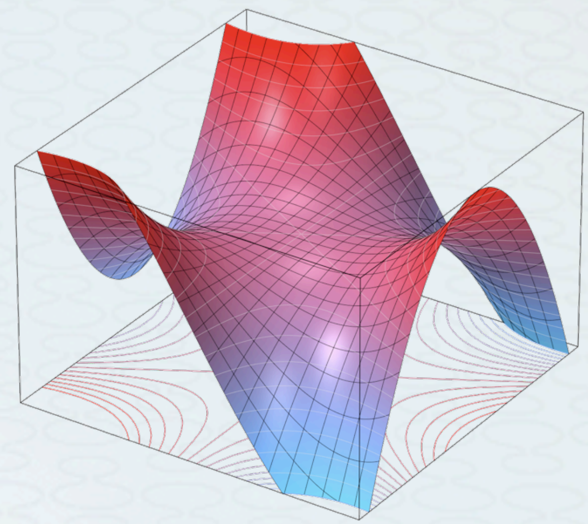
The Index Determines the Type of a Critical Point

- Index = number of negative Eigenvalues
- All positive (index=0): Minimum or valley
- All negative (index=dimension): Maximum or peak
- Otherwise: Saddle or pass (ambiguous point)



Simple Saddle

(images from wikipedia)



Monkey Saddle

Morse Functions

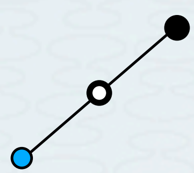
- Smooth (C^2-continuous) function
- Critical points are *non-degenerate*
  - Hessian non-singular (i.e., non-zero determinant)

  - No monkey saddles
- Critical points have distinct function values, i.e.,

$$\mathbf{p} \neq \mathbf{q} \Rightarrow f(\mathbf{p}) \neq f(\mathbf{q})$$

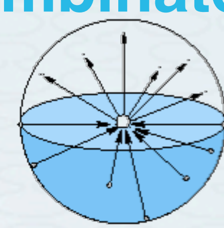
Combinatorial Definitions Provide a Robust Way to Find Critical Points

	$f(x): D \rightarrow \Re$ Classical Mathematical Definitions	$F(x): S \rightarrow \Re$ Simulation of Differentiability
domain	D smooth manifold	S simplicial complex
function	f Infinitely differentiable	$F(x)$ PL-extension of $f(x_i)$
critical point	$\nabla f(p) = 0$ **numerical**	$LowerLink(p) \neq B^{d-1}$ **combinatorial**



1D

2D



3D

Independent local computation yields globally consistent results

Combinatorial Criteria Count Positive and Negative Regions in Neighborhood

type	index
Minimum	0
⋮	⋮
Saddle	d-1
Maximum	*d*

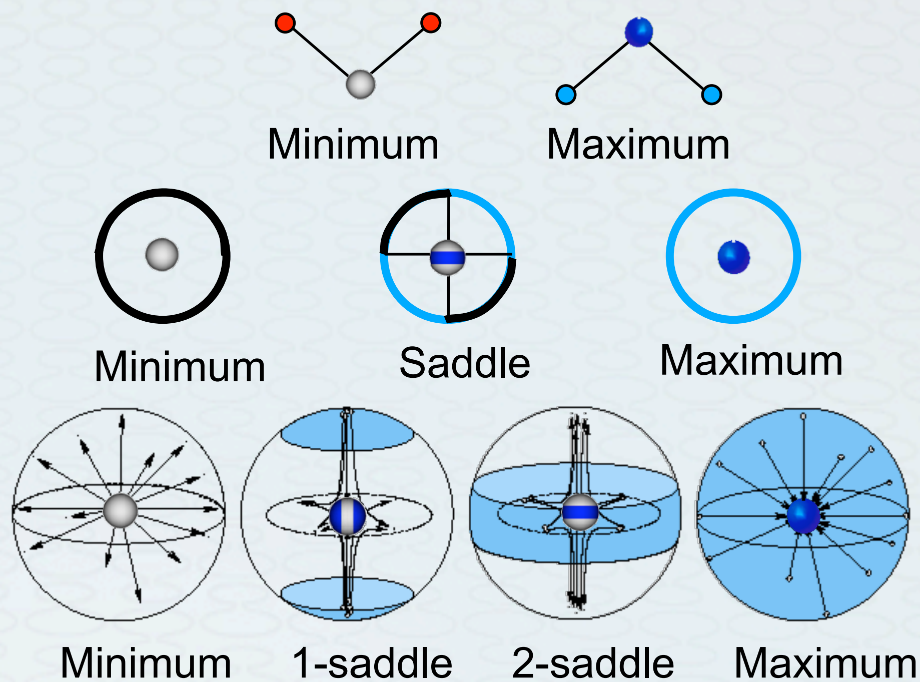
$$\nabla f(p) = 0$$

$$f(x)|_p = f(p) + \sum_{i=1}^{d-k} x_i^2 - \sum_{j=d-k+1}^{d} x_j^2$$

numerical

The Morse Lemma

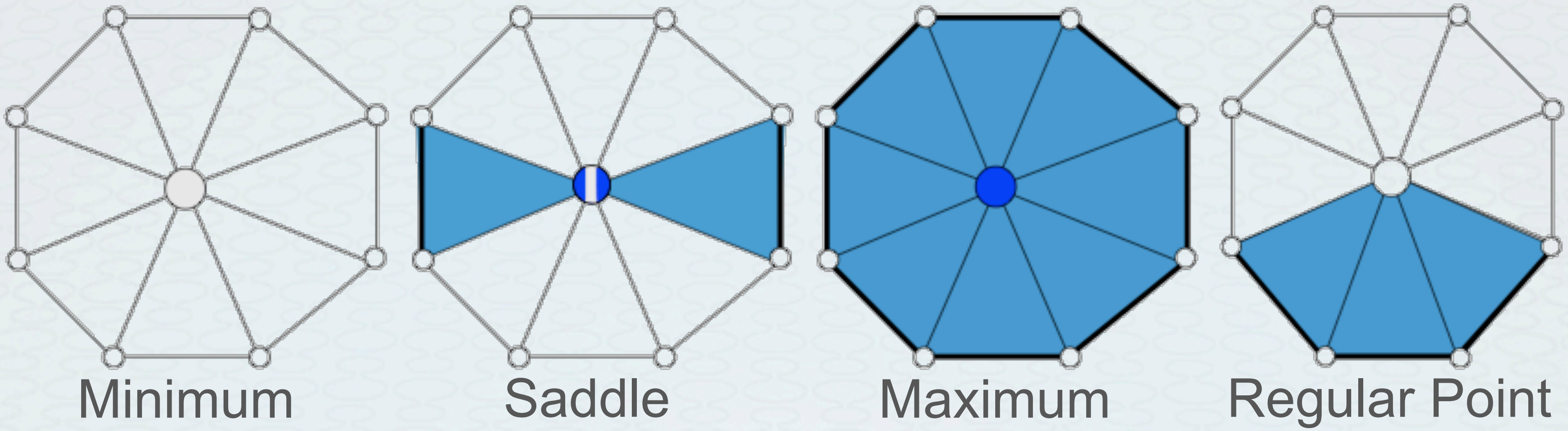
There are *d+1* types
of critical points



combinatorial

Detecting Critical Points for Piecewise Linear Functions on a Simplicial Complex

- Example: Points in 2D – Triangulation
- Classify point by considering its neighborhood [Banchoff 1970/83], [Edelsbrunner et al., 2003]



- 3D Analogous, see [Edelsbrunner et al., Proc. 19th Ann., 2003]

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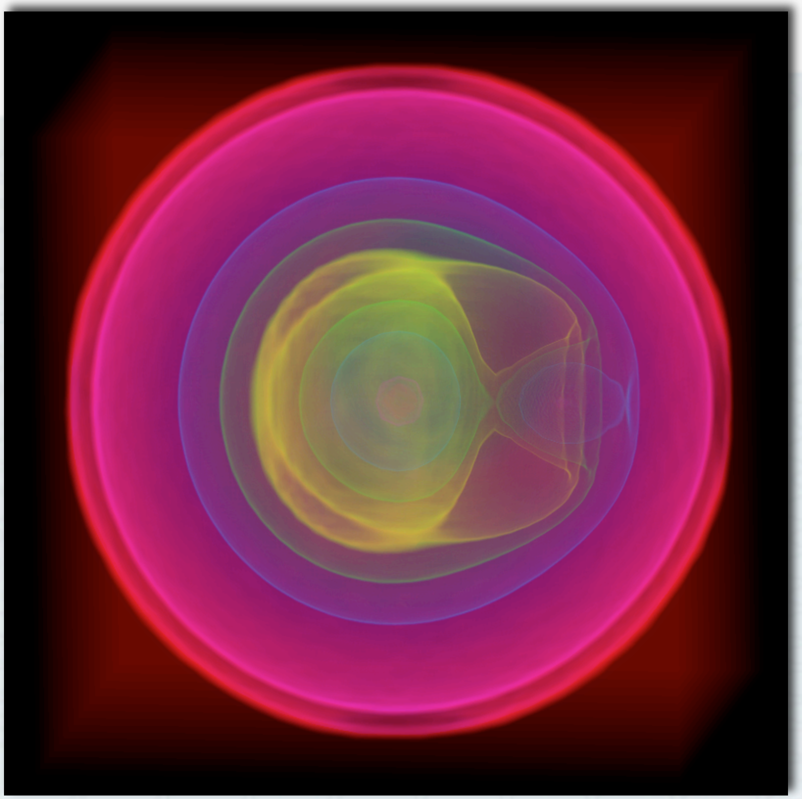
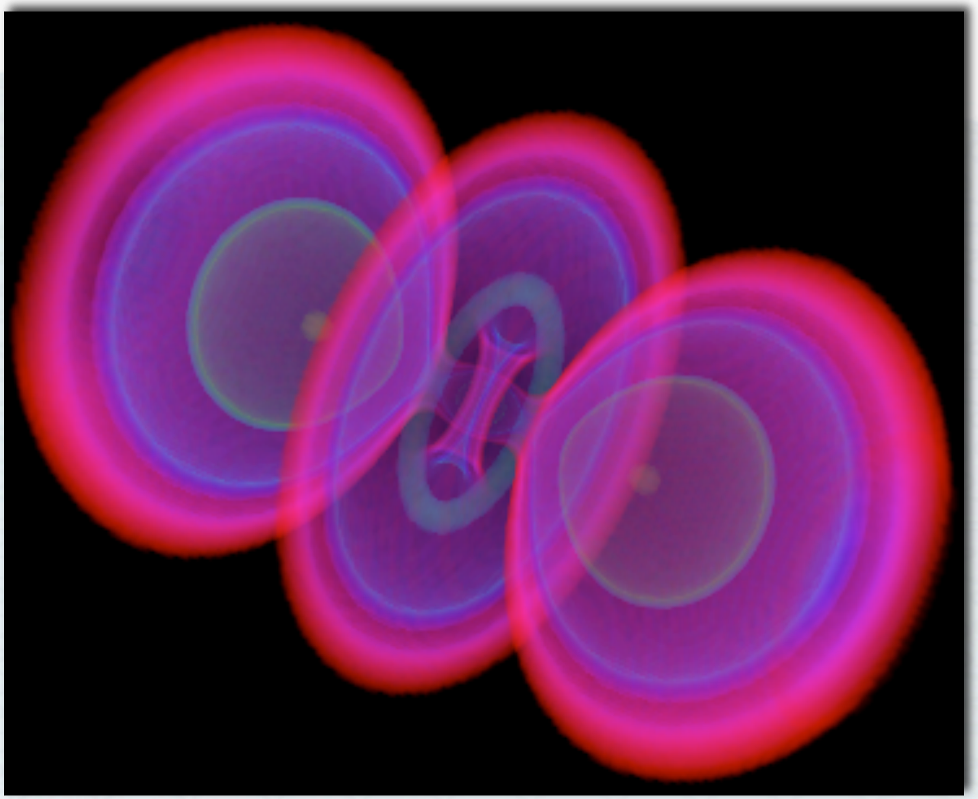
Critical Points Can Help in Identifying Relevant Isosurfaces

- Scalar topology reveals hidden isosurface component in the probability distribution of the location of a nucleon.
  [Fujishiro et al., IEEE CG&A 2000], [Weber et al., IEEE Visualization 2002 Conference], [Weber et al., Eurographics/IEEE ViSym 2003] (Dataset: SFB 382, DFG)

Critical Points Define Transfer Functions Emphasizing Topological Changes





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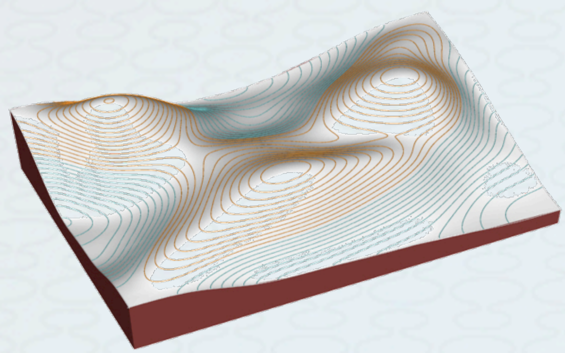
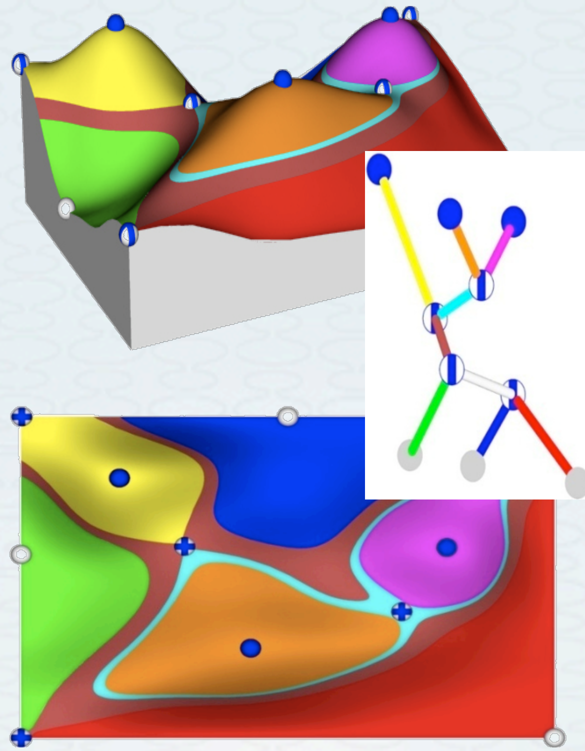
Topological Structures Define Relationship Between Critical Points

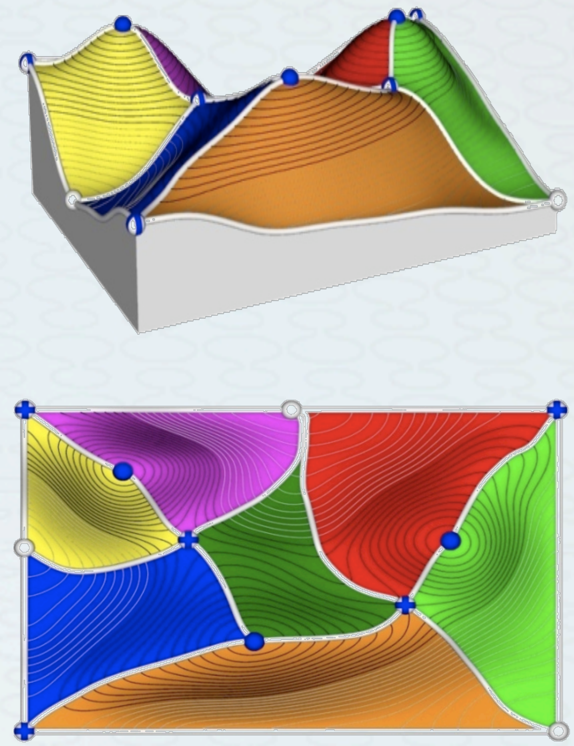
- Describe “feature space”
- Simplification and data/dimensionality reduction



Reeb graph/ contour tree



Morse-Smale complex



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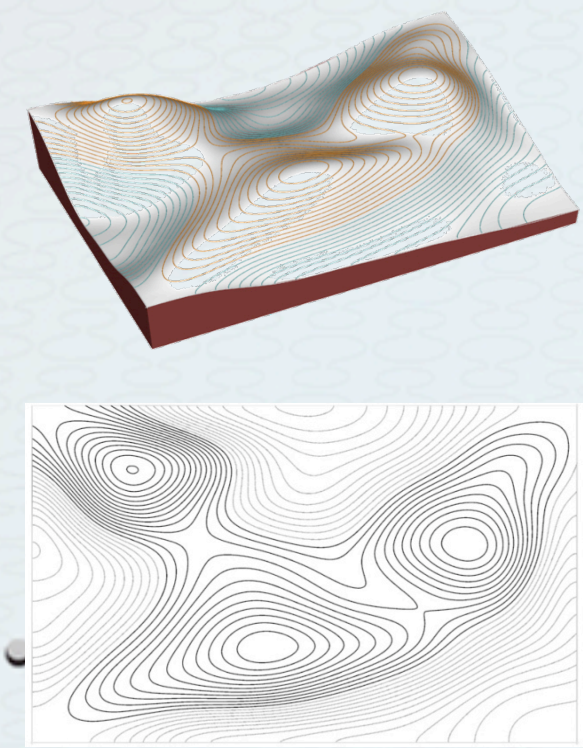

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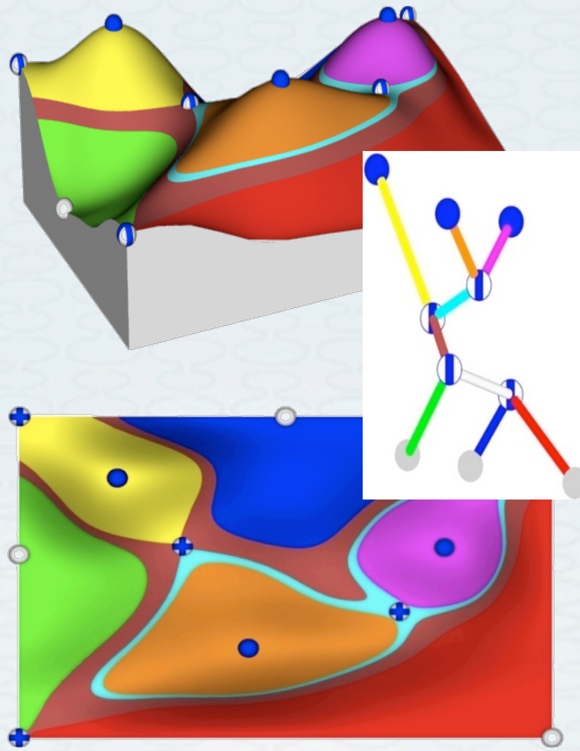

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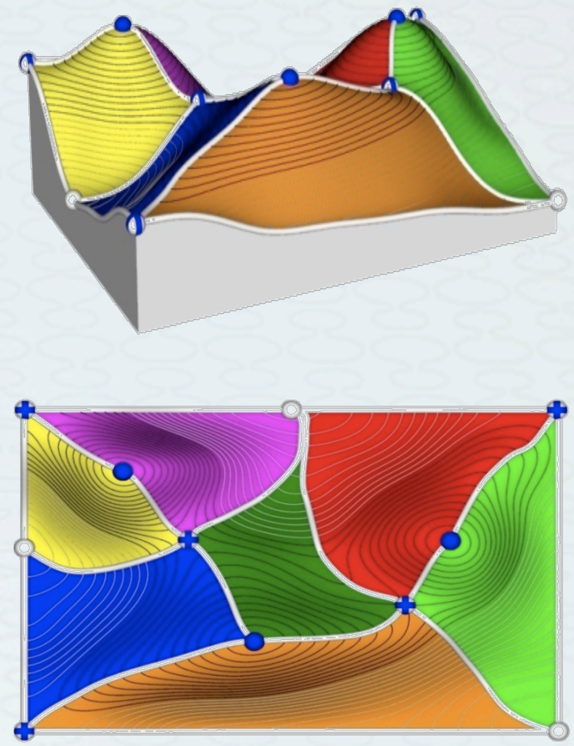
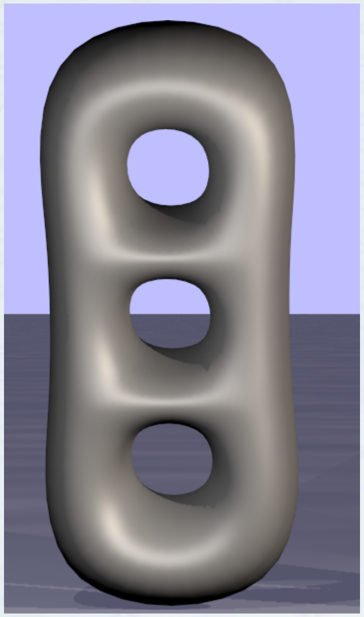


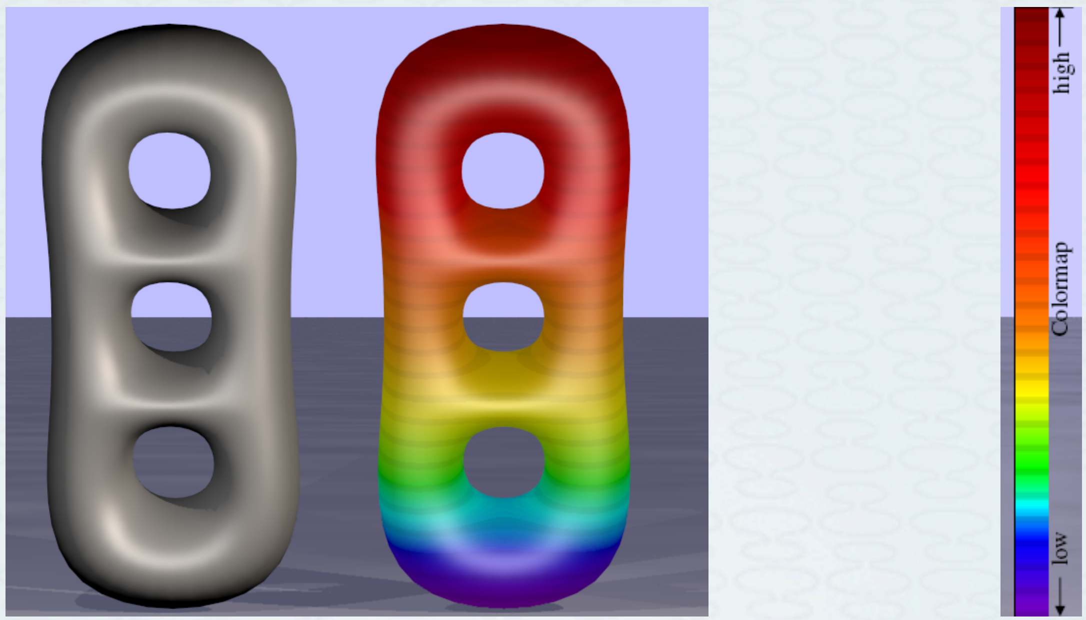
The Reeb Graph Is the Contraction of Isocontour Components to Points

- Given a mesh.



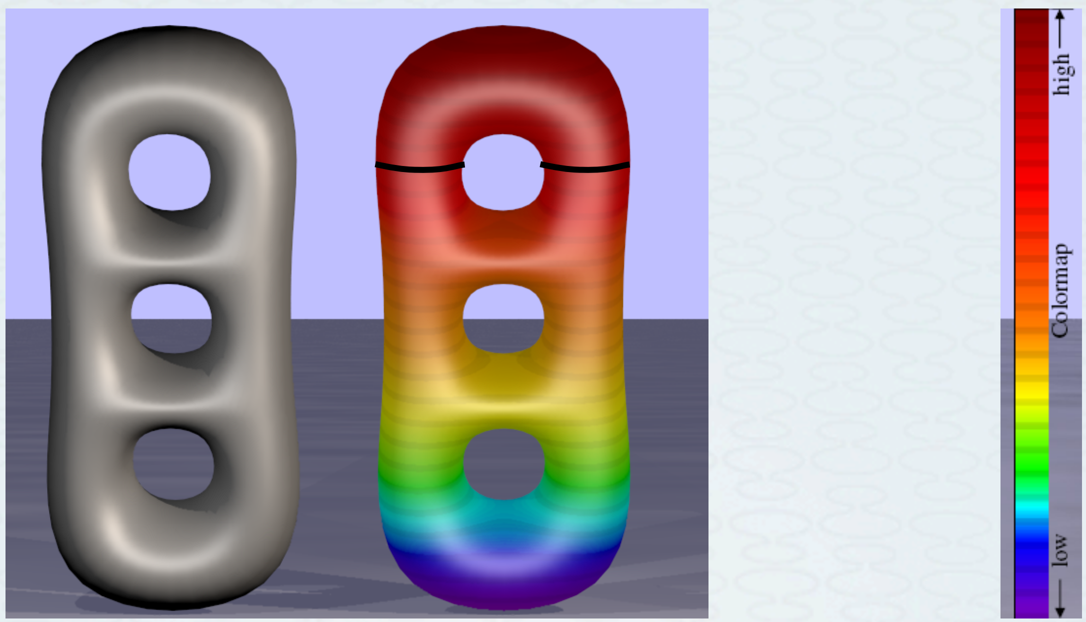
The Reeb Graph Is the Contraction of Isocontour Components to Points

- Given a mesh and a function defined on it.



The Reeb Graph Is the Contraction of Isocontour Components to Points

- Given a mesh and a function defined on it.
- Consider an isocontour.



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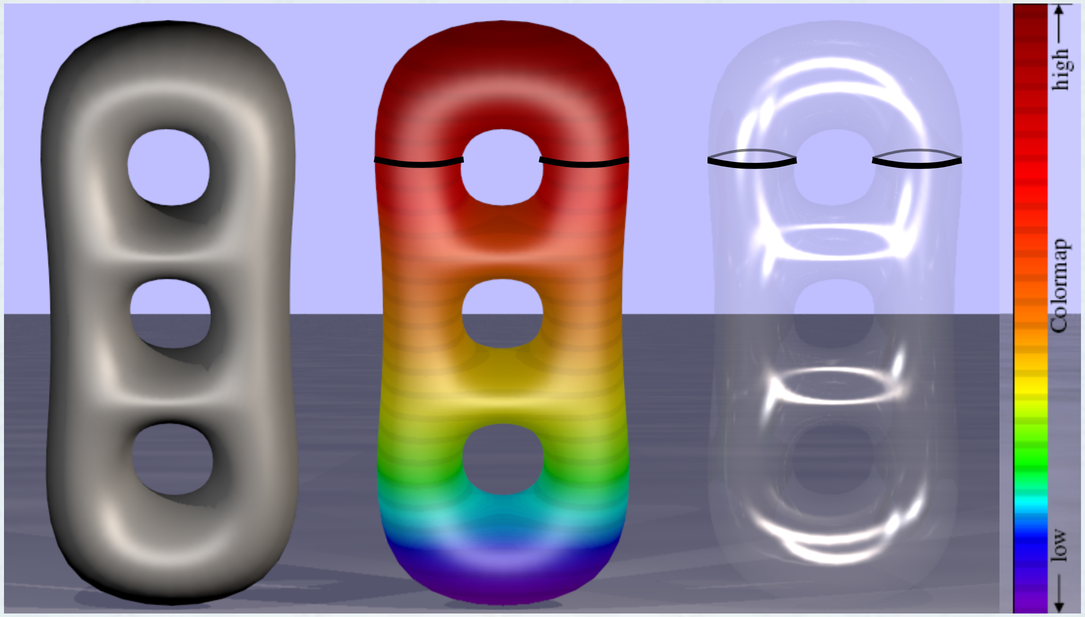

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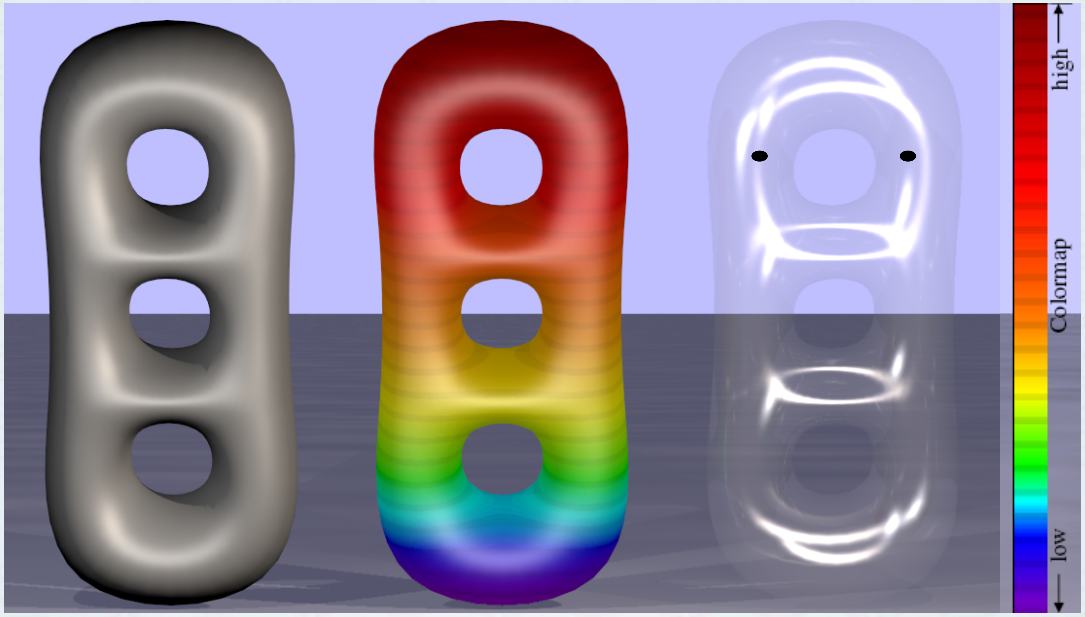
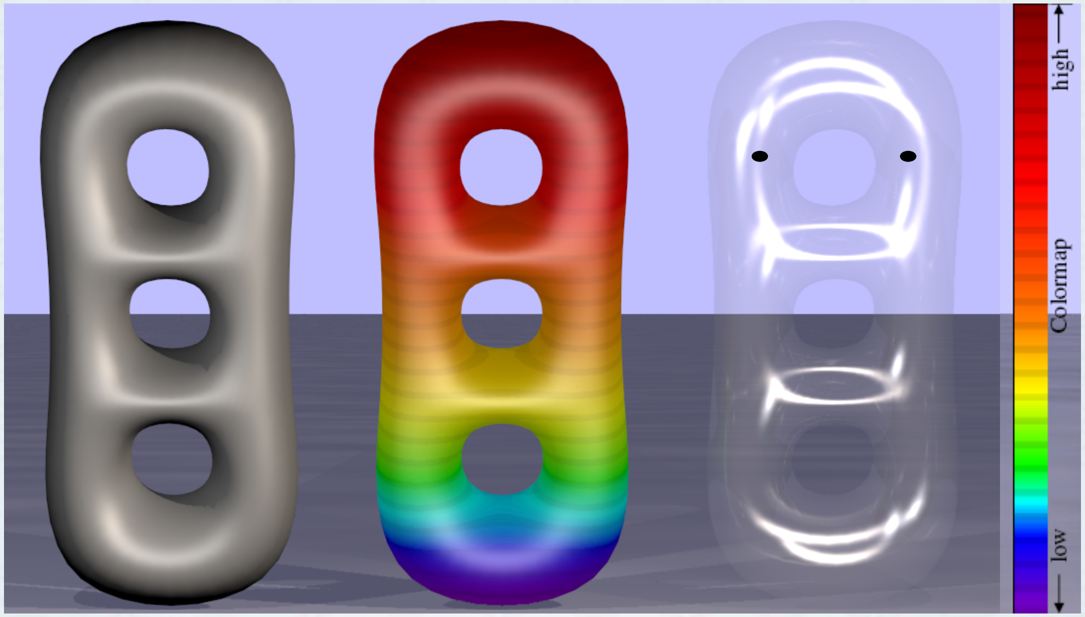
The Reeb Graph Is the Contraction of Isocontour Components to Points

- Given a mesh and a function defined on it.
- Consider an isocontour and contract each component.

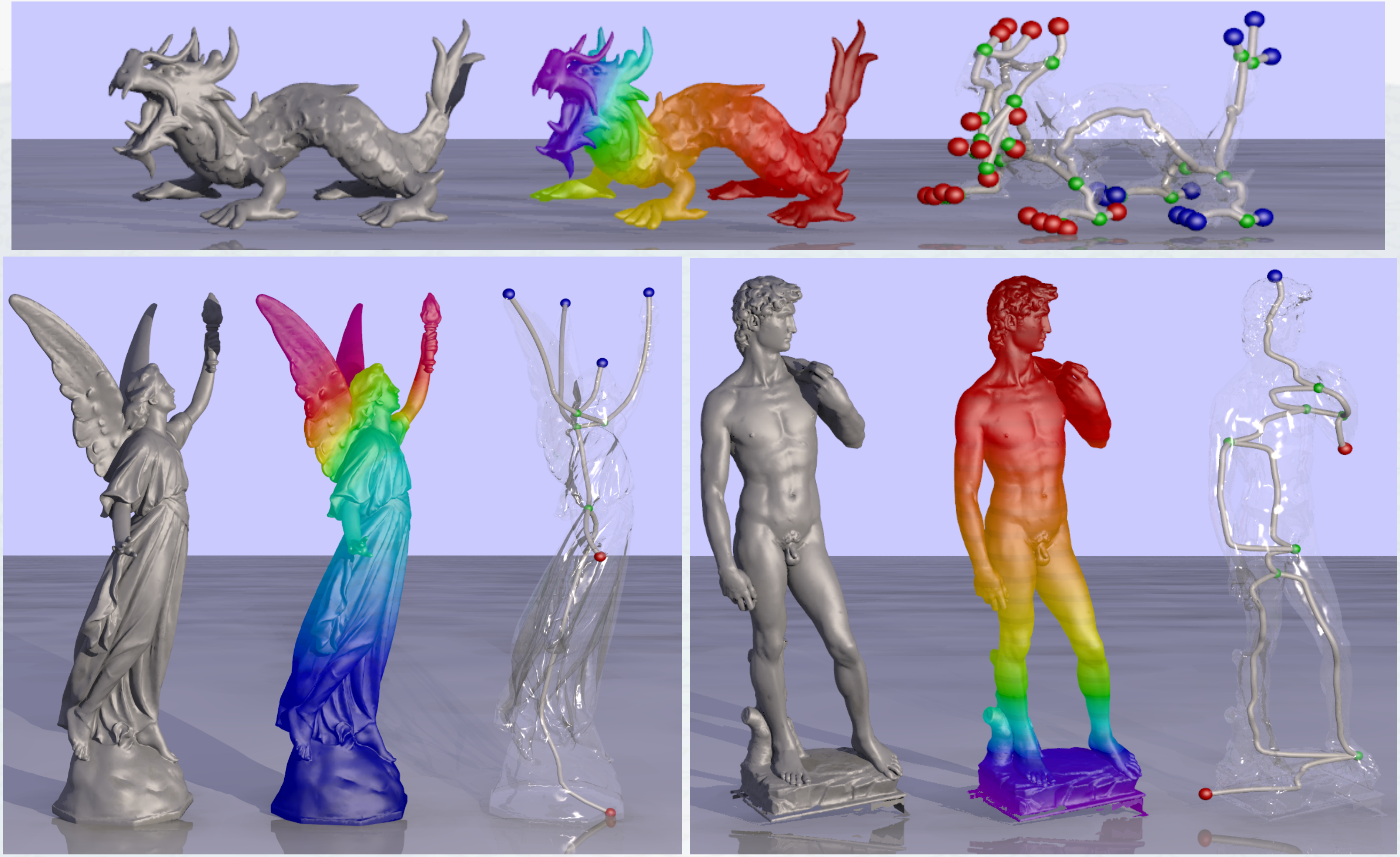


The Reeb Graph Is the Contraction of Isocontour Components to Points

- Given a mesh and a function defined on it.
- Consider an isocontour and contract each component.
- Repeat for all contours while maintaining adjacency.

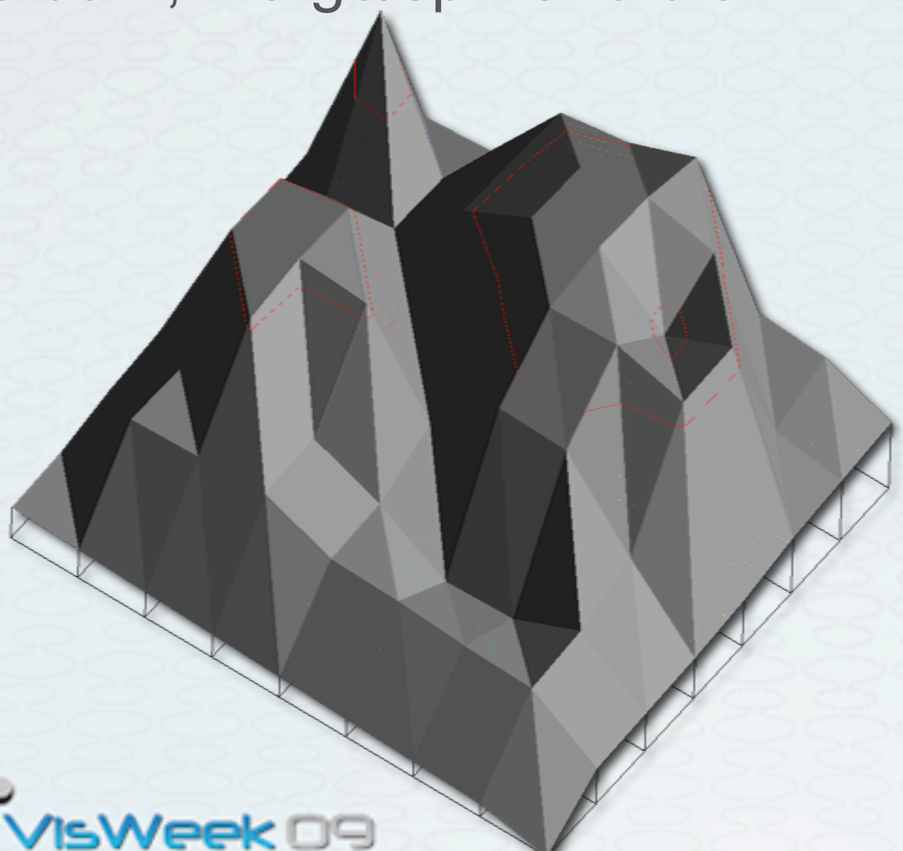


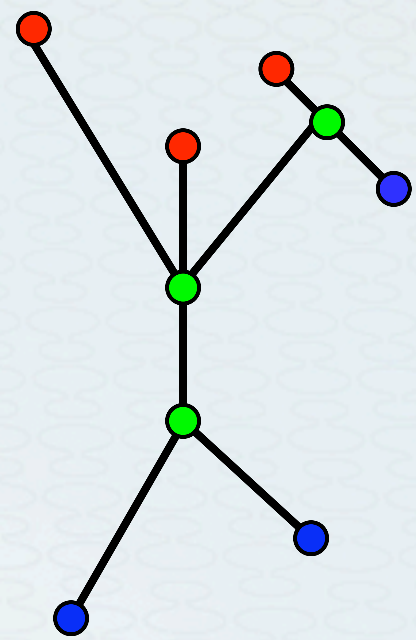
The Reeb Graph Represents the Skeleton of a Shape



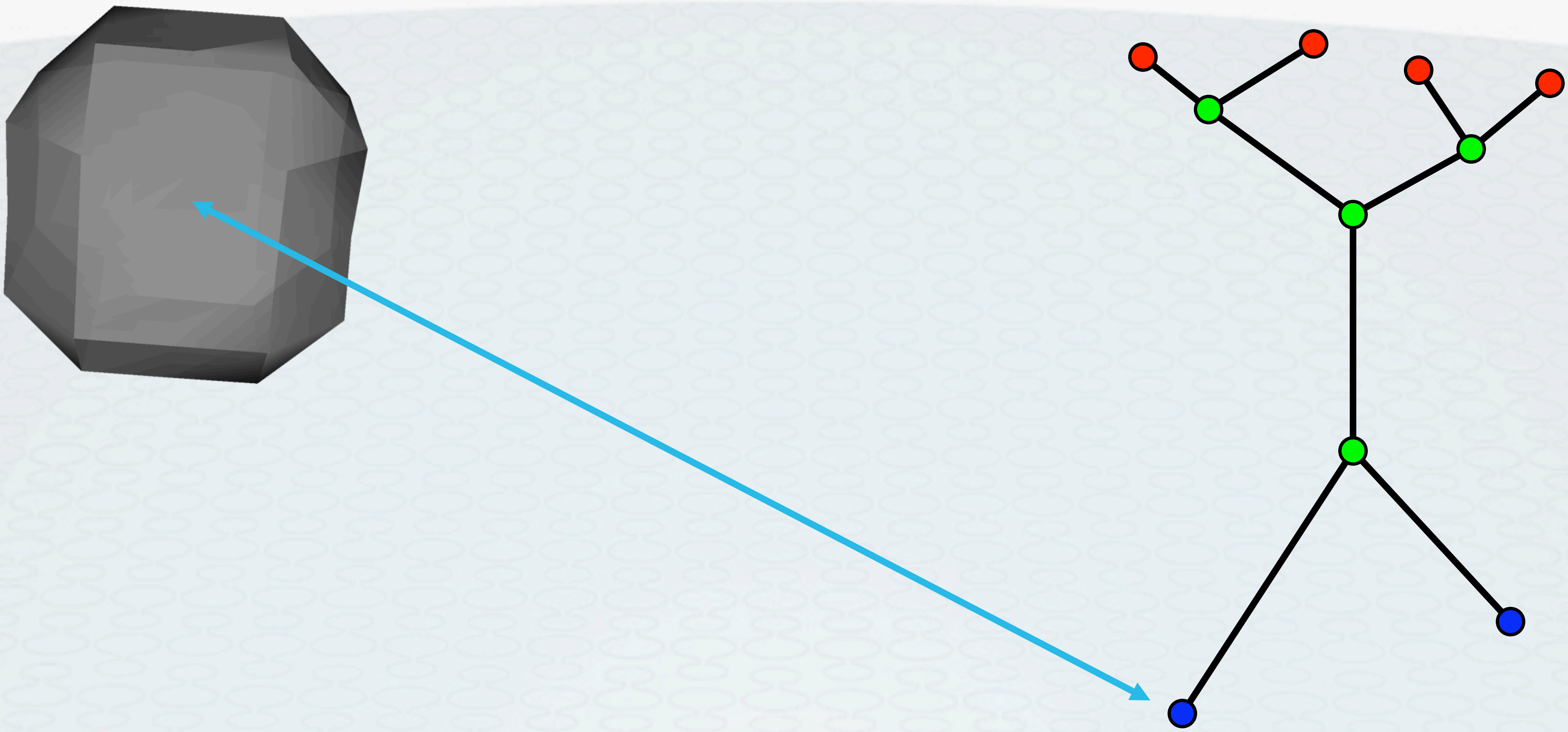
Contour Tree

- Simply connected domain ➔ General graph becomes tree
- Tracks contours (connected isosurface components) as they are born, merge/split and die





Contour Tree – Example



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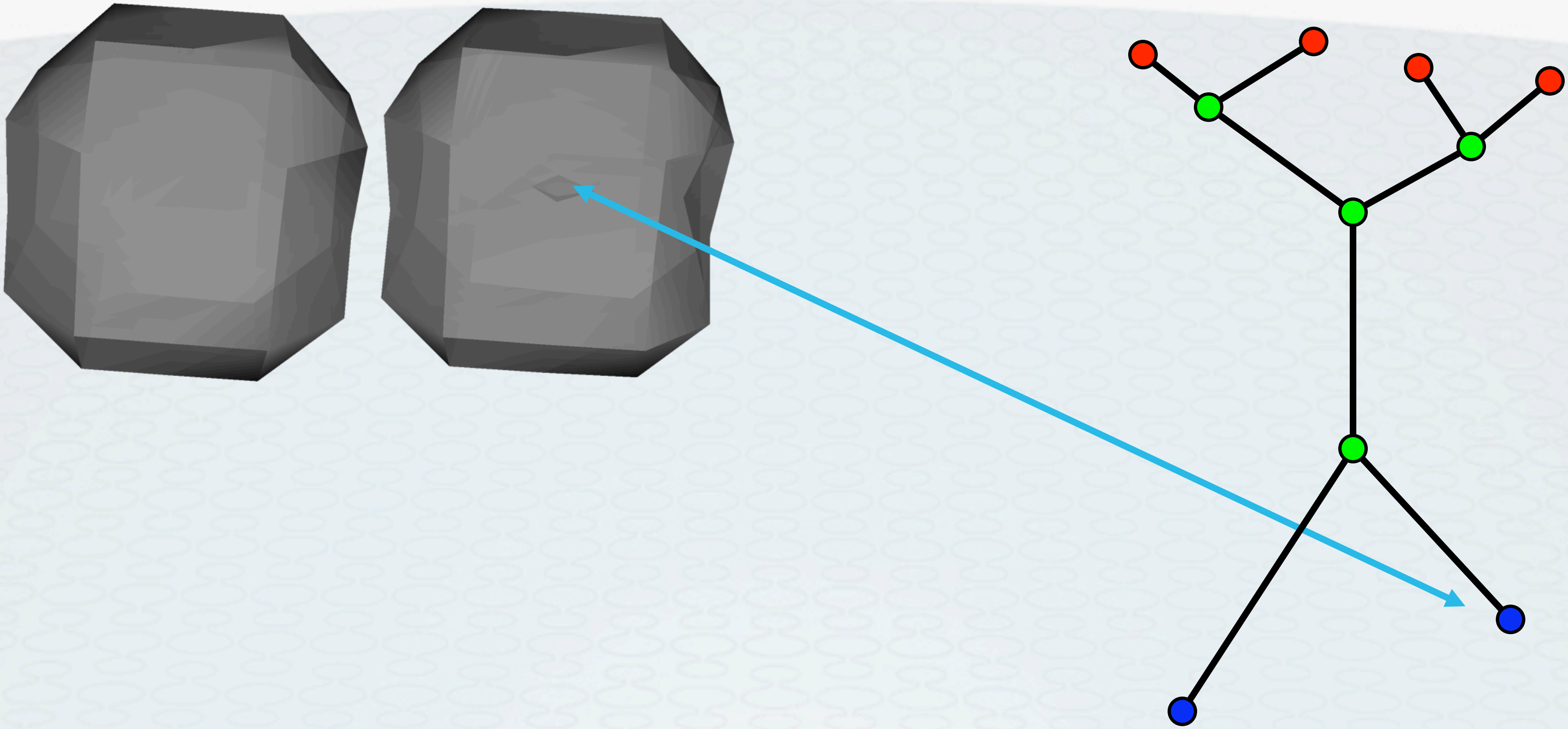

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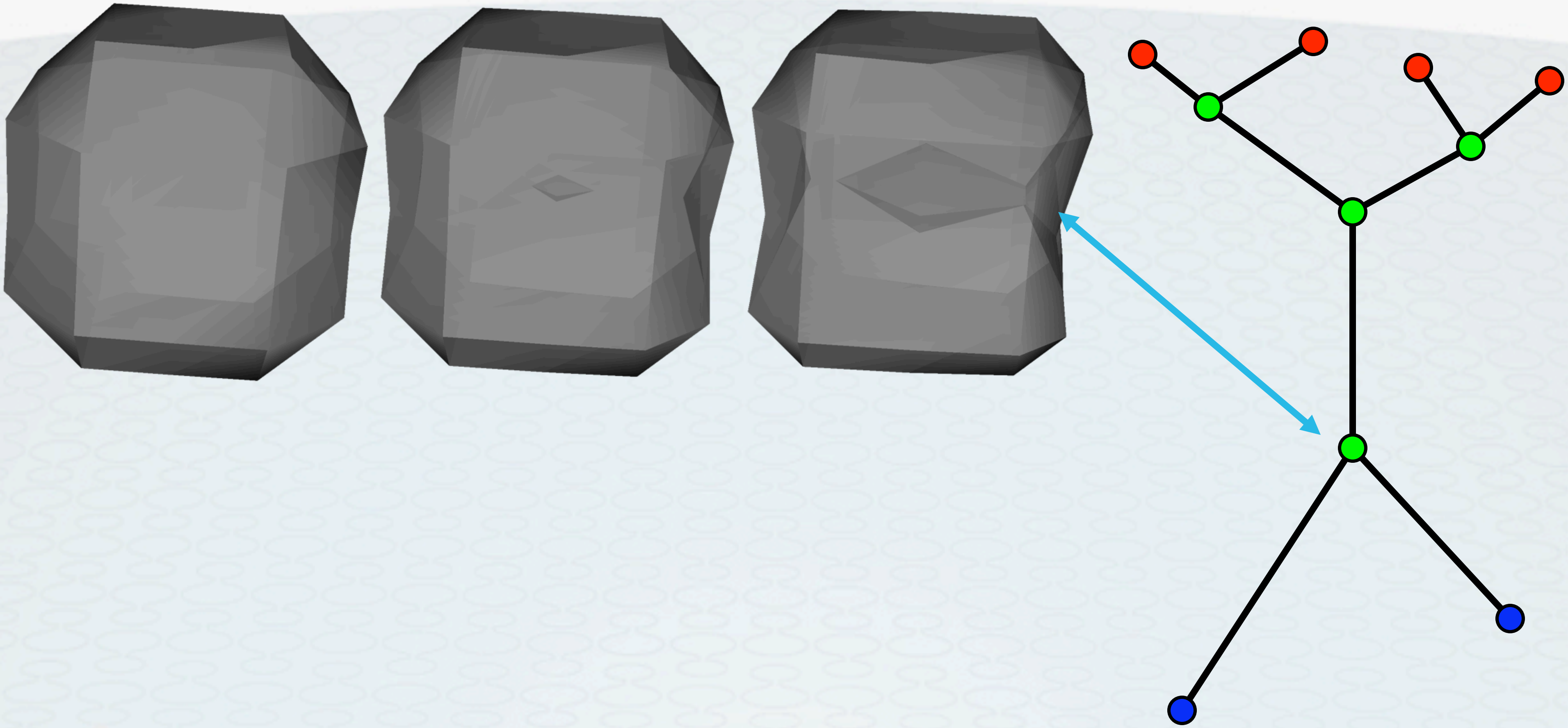
Contour Tree – Example



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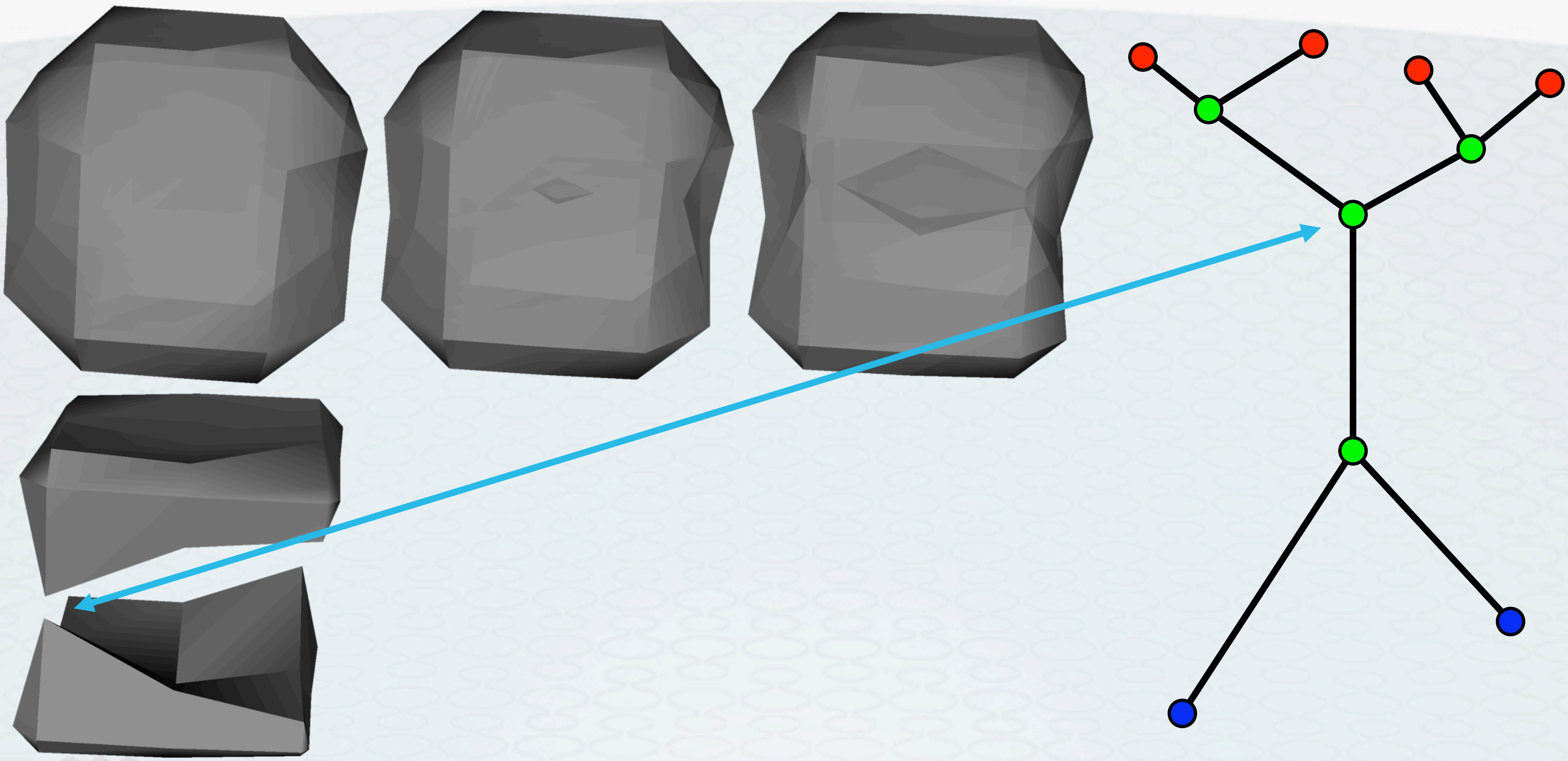

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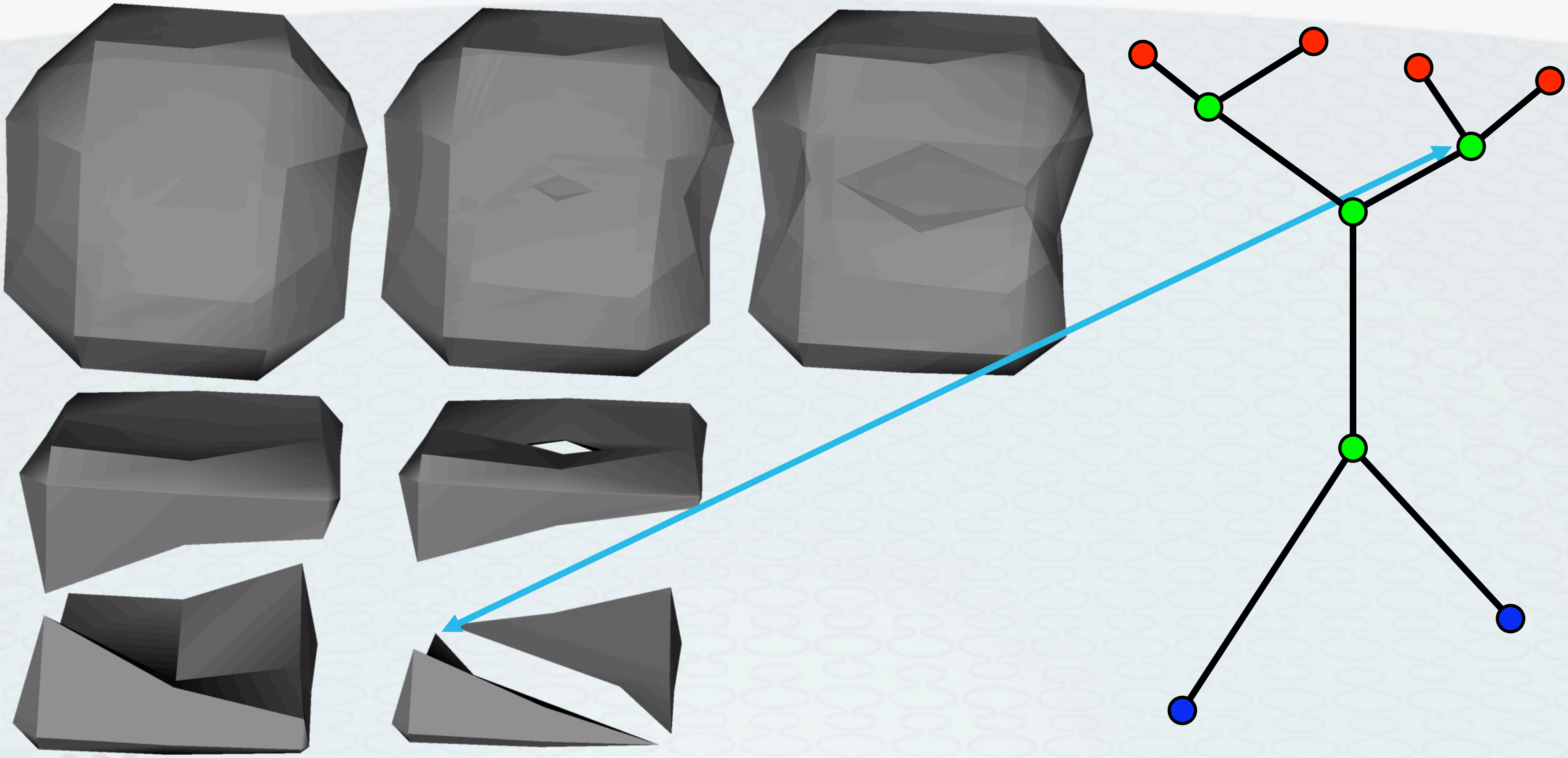

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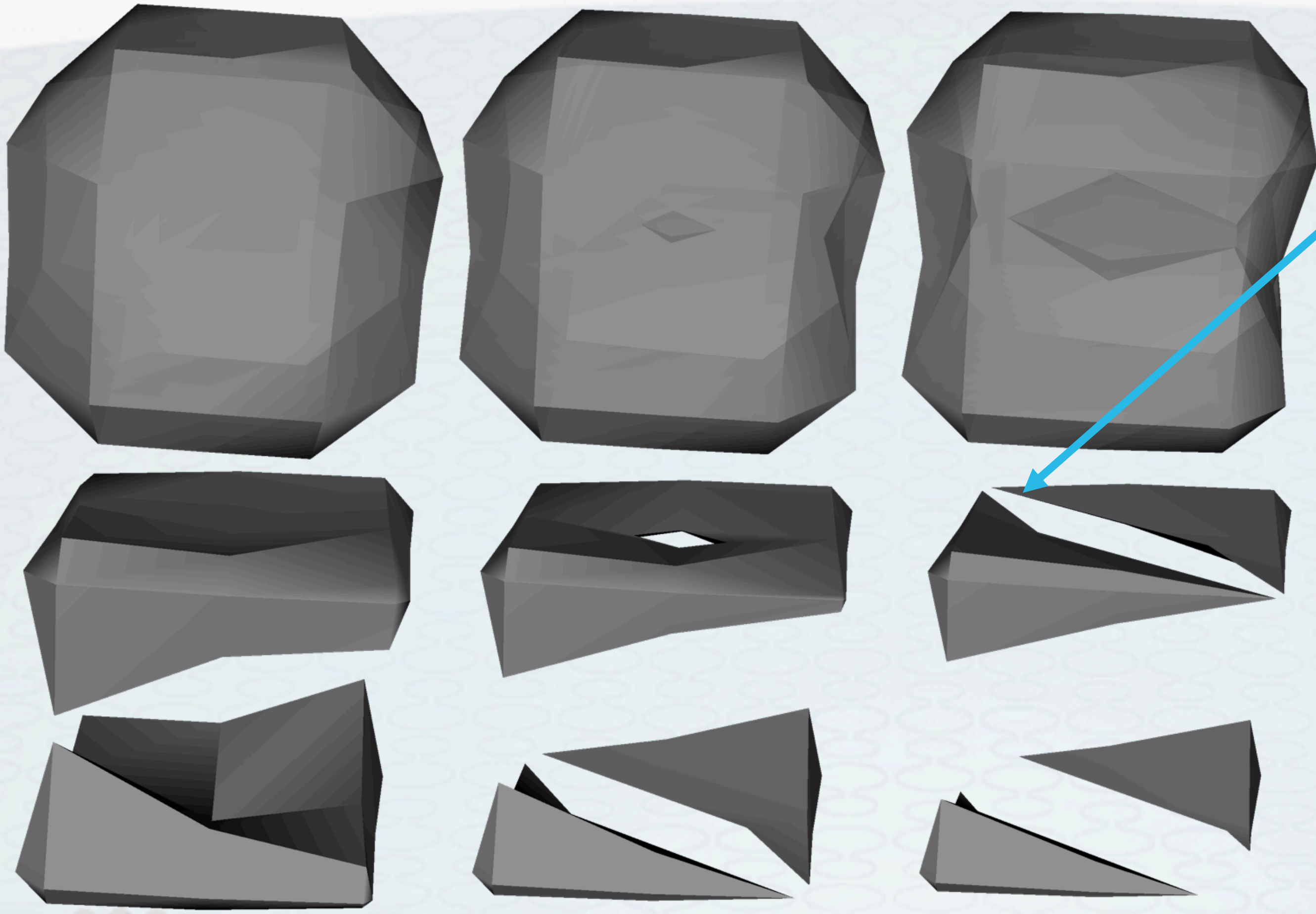
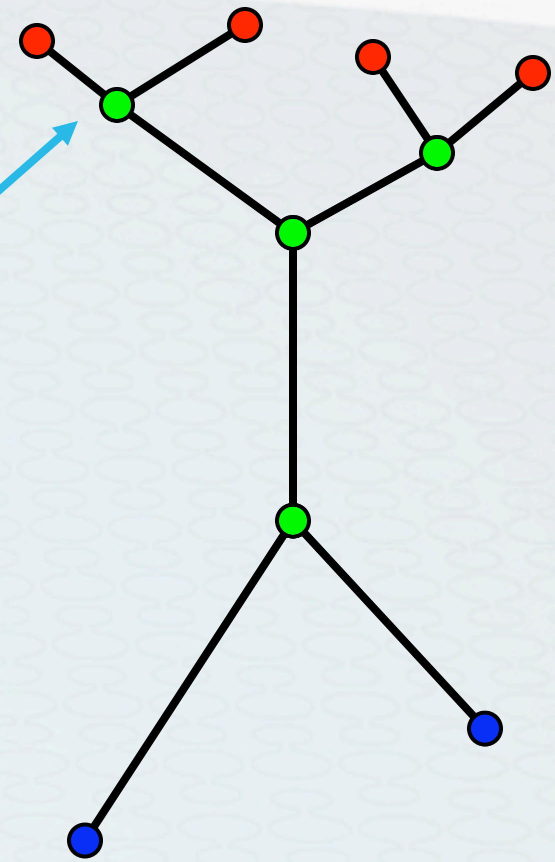
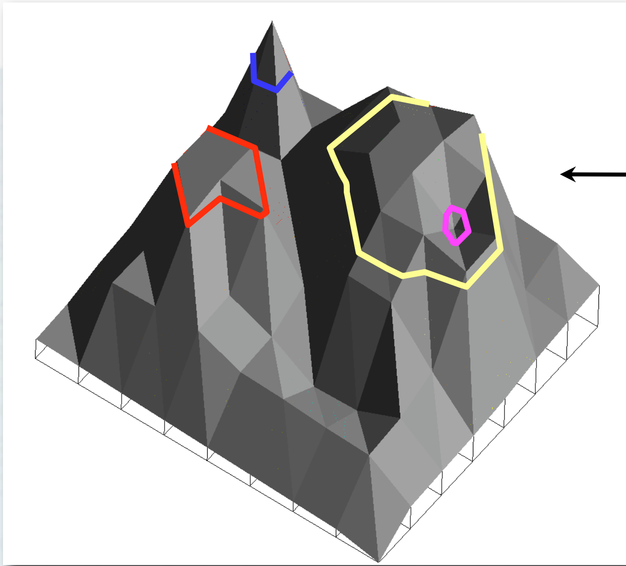
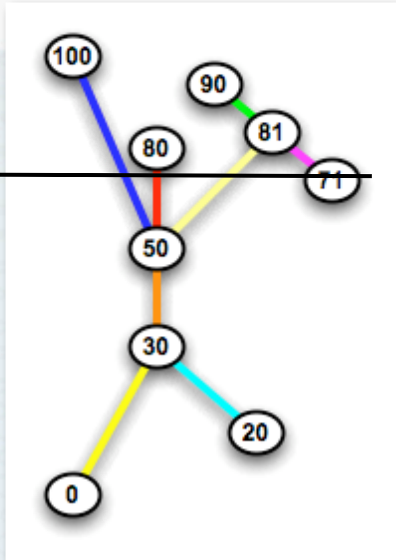




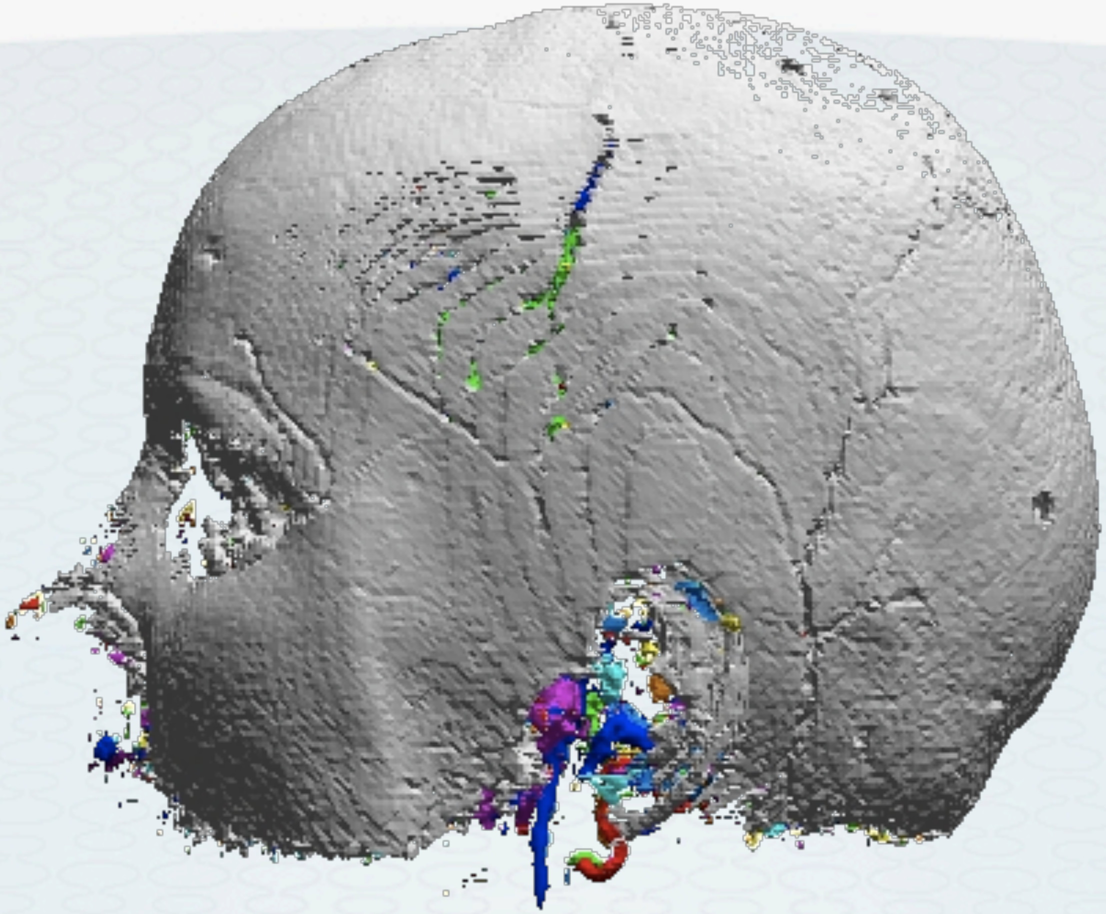
Applications of the Contour Tree – Flexible Isosurfaces





- Speed-up of isosurface extraction by finding minimal seed-sets for continuation method
- “Flexible isosurfaces”: Contours (connected components) as individual entities (Carr et al., 2003)

Flexible Isosurfaces Examples – Remove Occluding Components



Flexible Isosurfaces Examples – Remove Occluding Components

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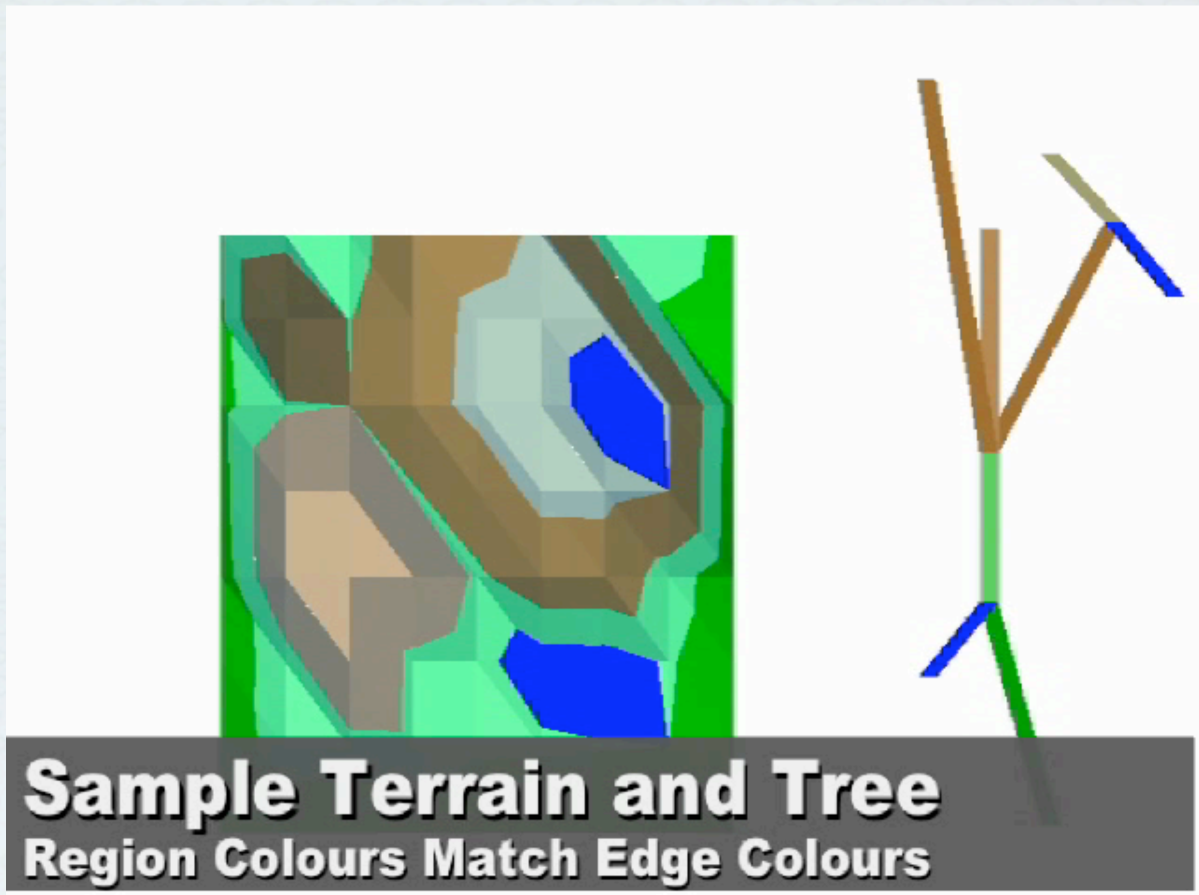
Flexible Isosurfaces Examples – Remove Occluding Components

Flexible Isosurfaces Examples – Remove Occluding Components

Complex Topology Necessitates Simplification Schemes

- Inherent data complexity
- Features at multiple scales
- Noise

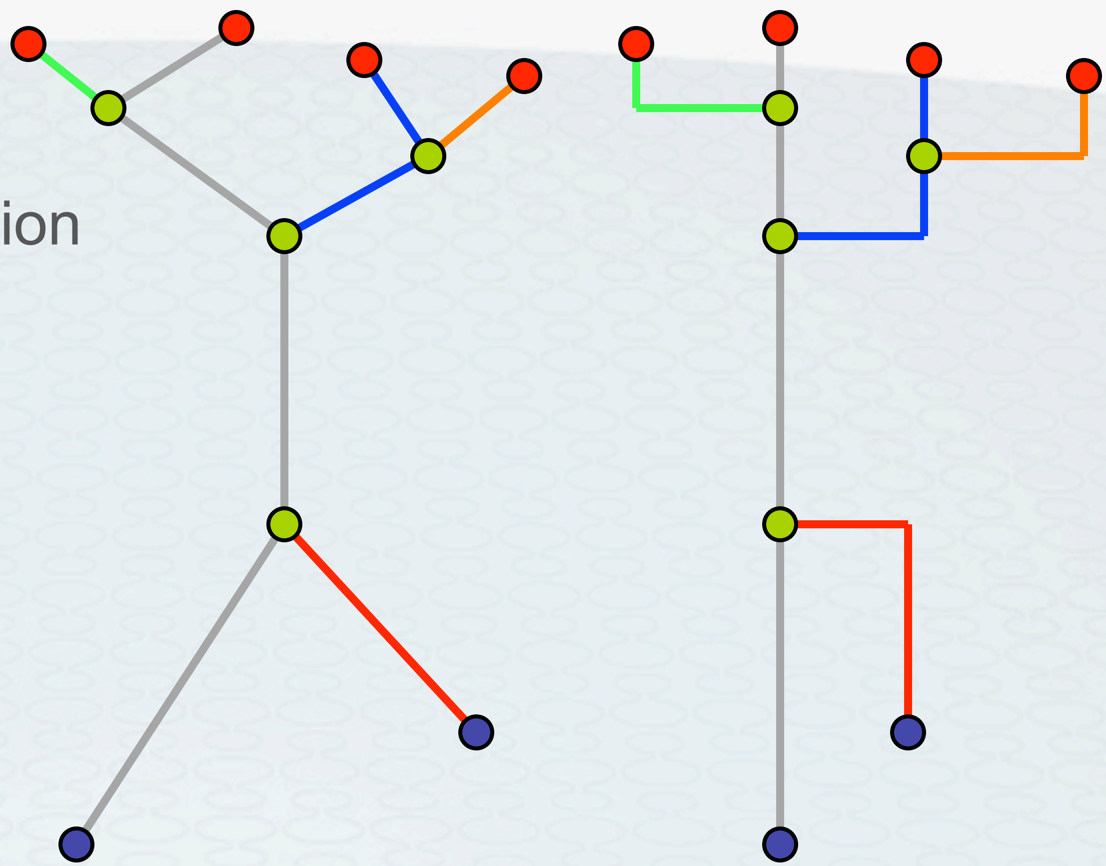


Sample Terrain and Tree
Region Colours Match Edge Colours

Branch Decomposition

➔ Hierarchical contour tree representation

- Order based on simplification measure, e.g.,
  - persistence
  - area/volume
  - hypervolume



(Pascucci et al., 2004)

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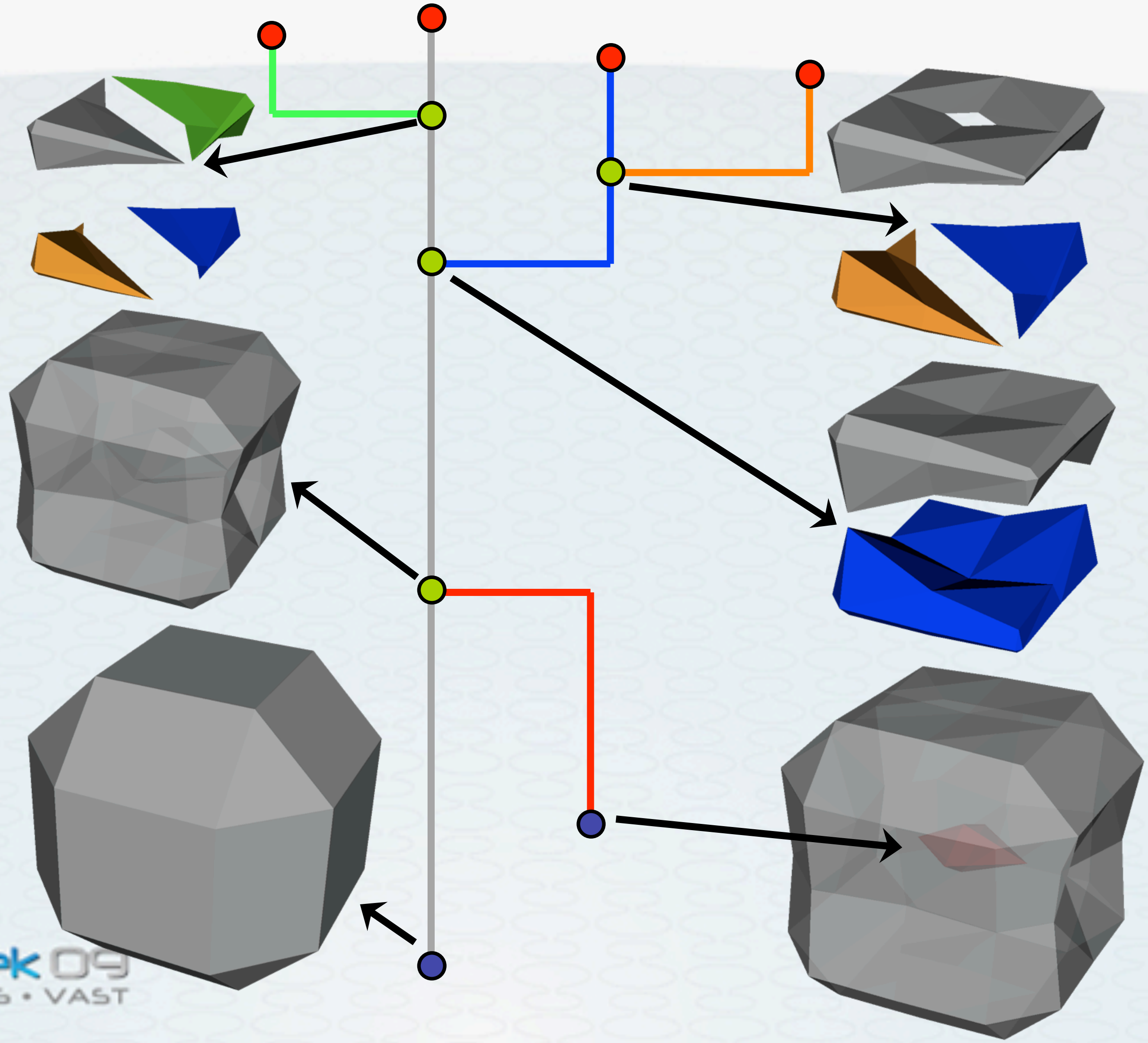
Branch Decomposition and Corresponding Contours



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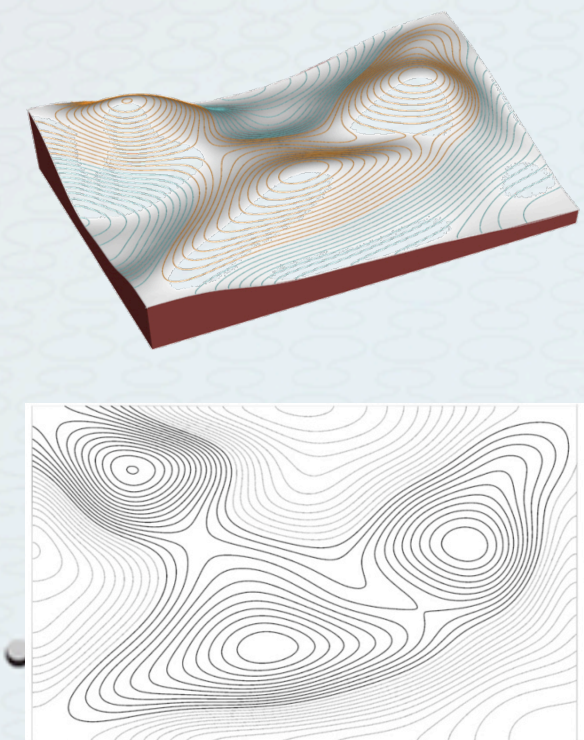
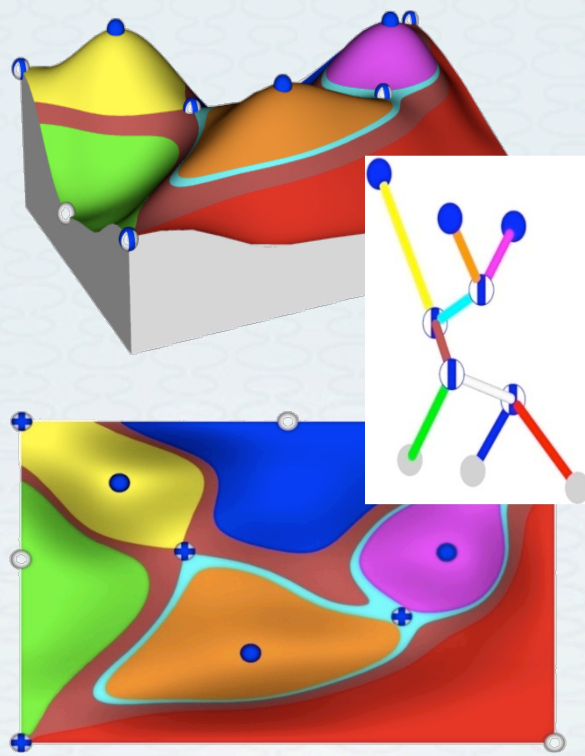
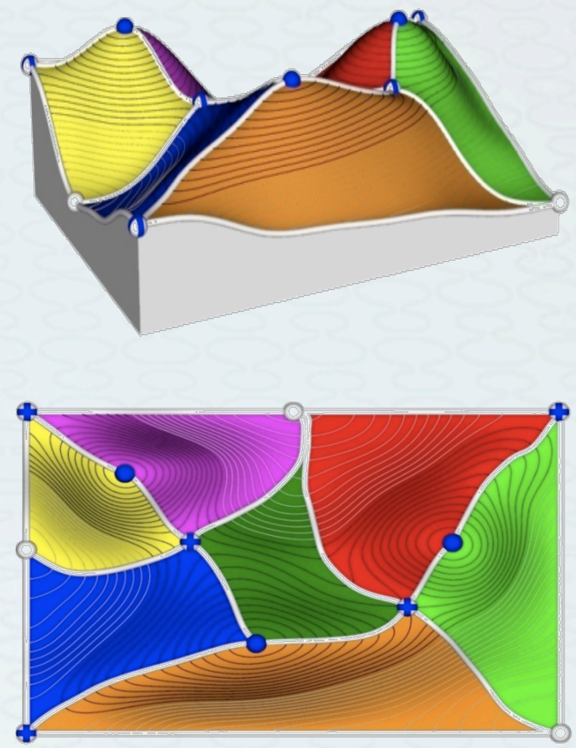
- Describe “feature space”
- Simplification and data/dimensionality reduction



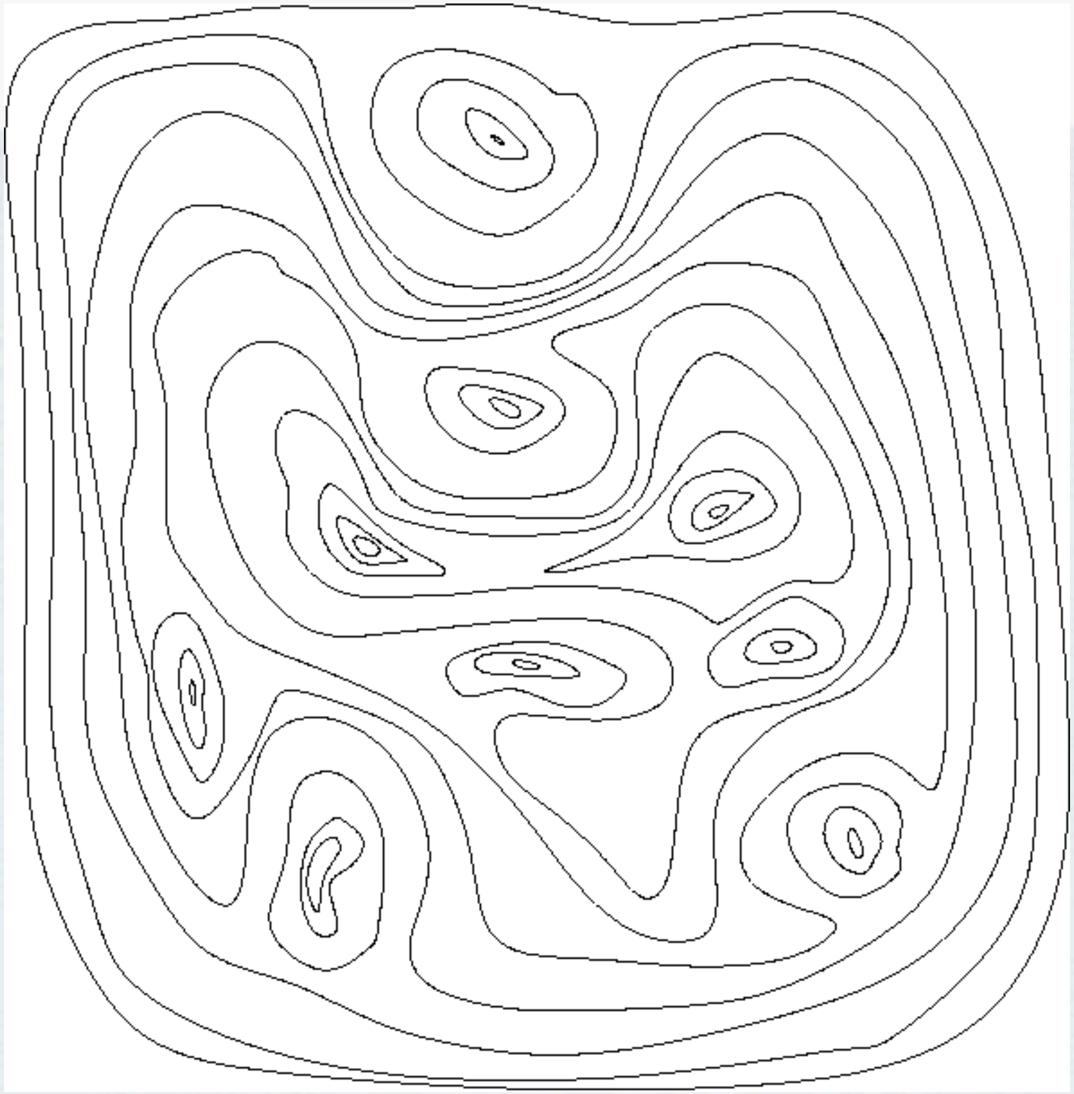
Reeb graph/ contour tree



Morse-Smale complex



Cayley (1859) / Maxwell (1870)

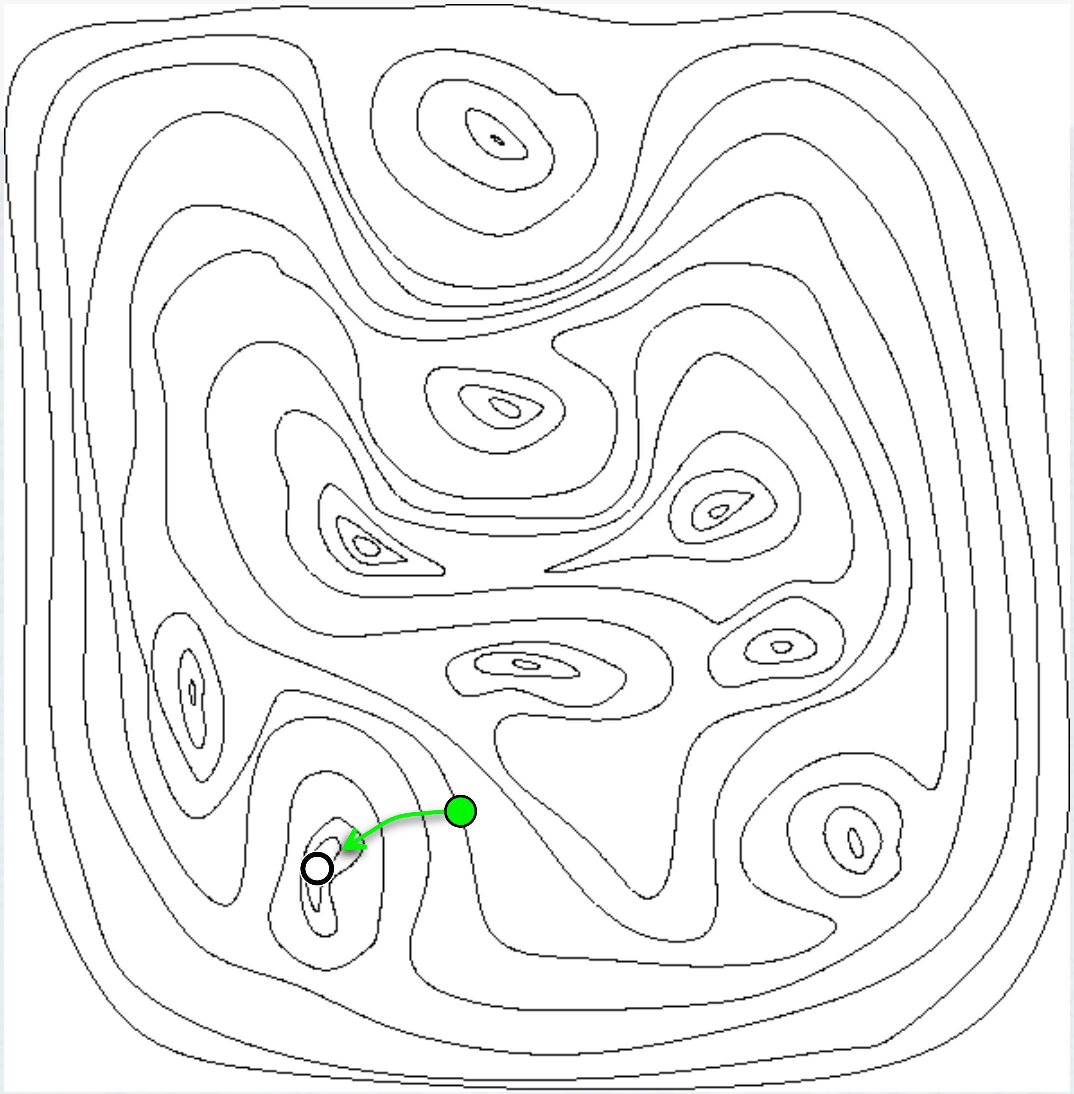


Gradient Lines

- Gradient indicates *steepest* ascent
- A *gradient line* runs from a minimum to a maximum
  - A maximal *path* $p : \mathbb{R} \to \mathbb{R}^n$
  - Such that $\frac{\delta}{\delta s} p(s) = \nabla f(p(s)) \forall s \in \mathbb{R}$
  - Paths are *monotone* between critical points
- All gradient lines
  - Start at minima or saddles
  - And lead to saddles or maxima

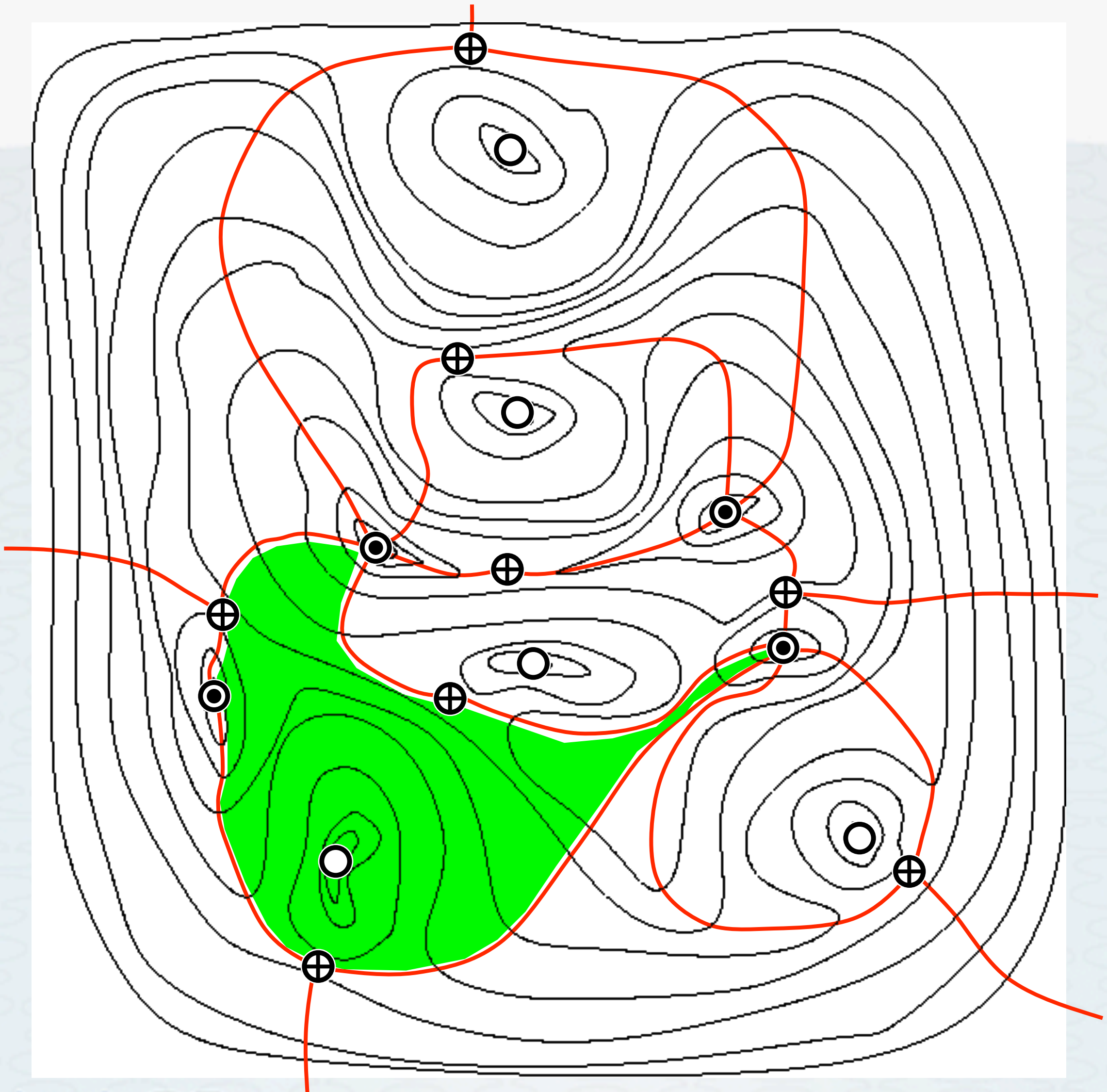
➜ Define equivalence between gradient lines based on start or end point

Lines of Steepest Descent



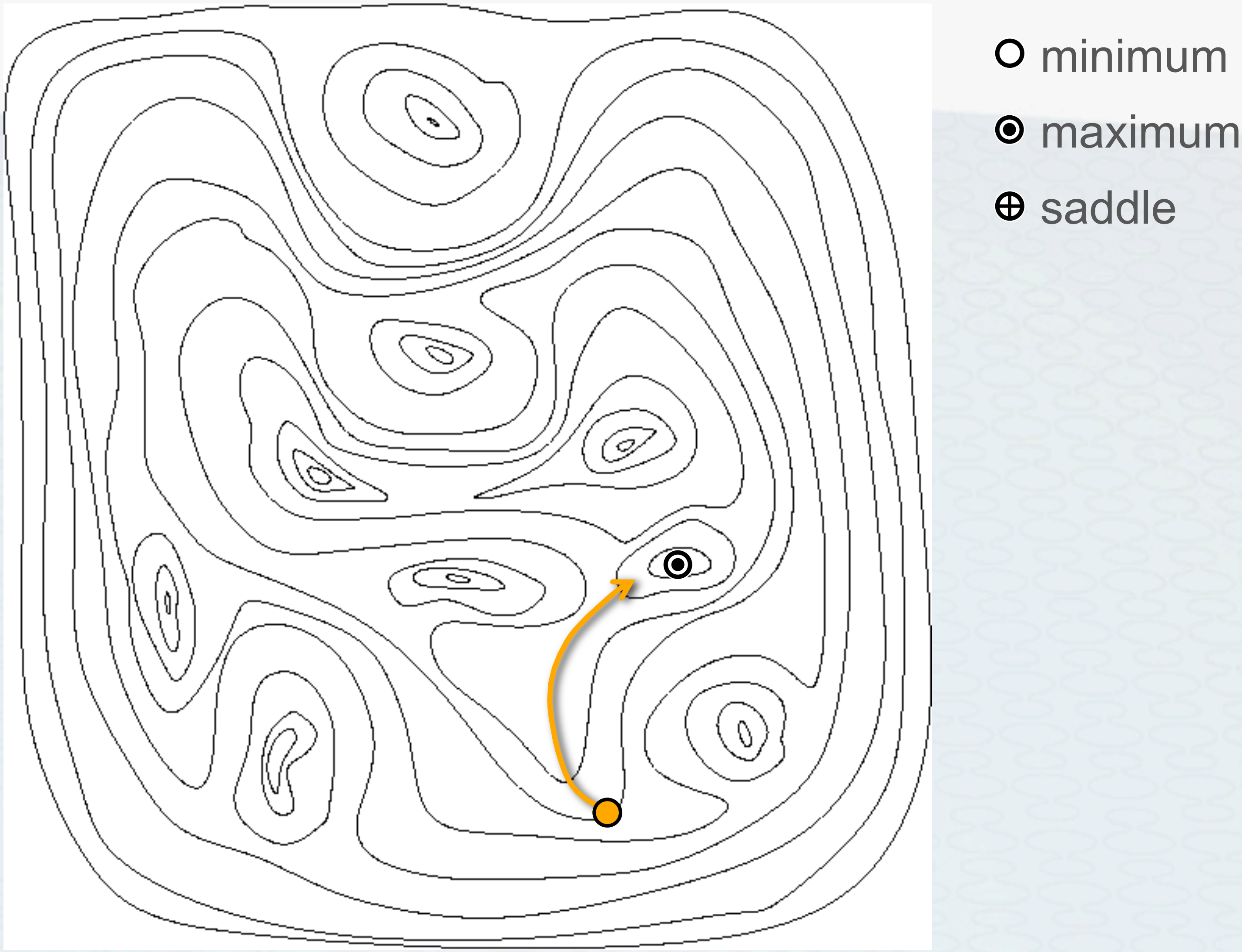
o minimum

Complex of Stable Manifolds

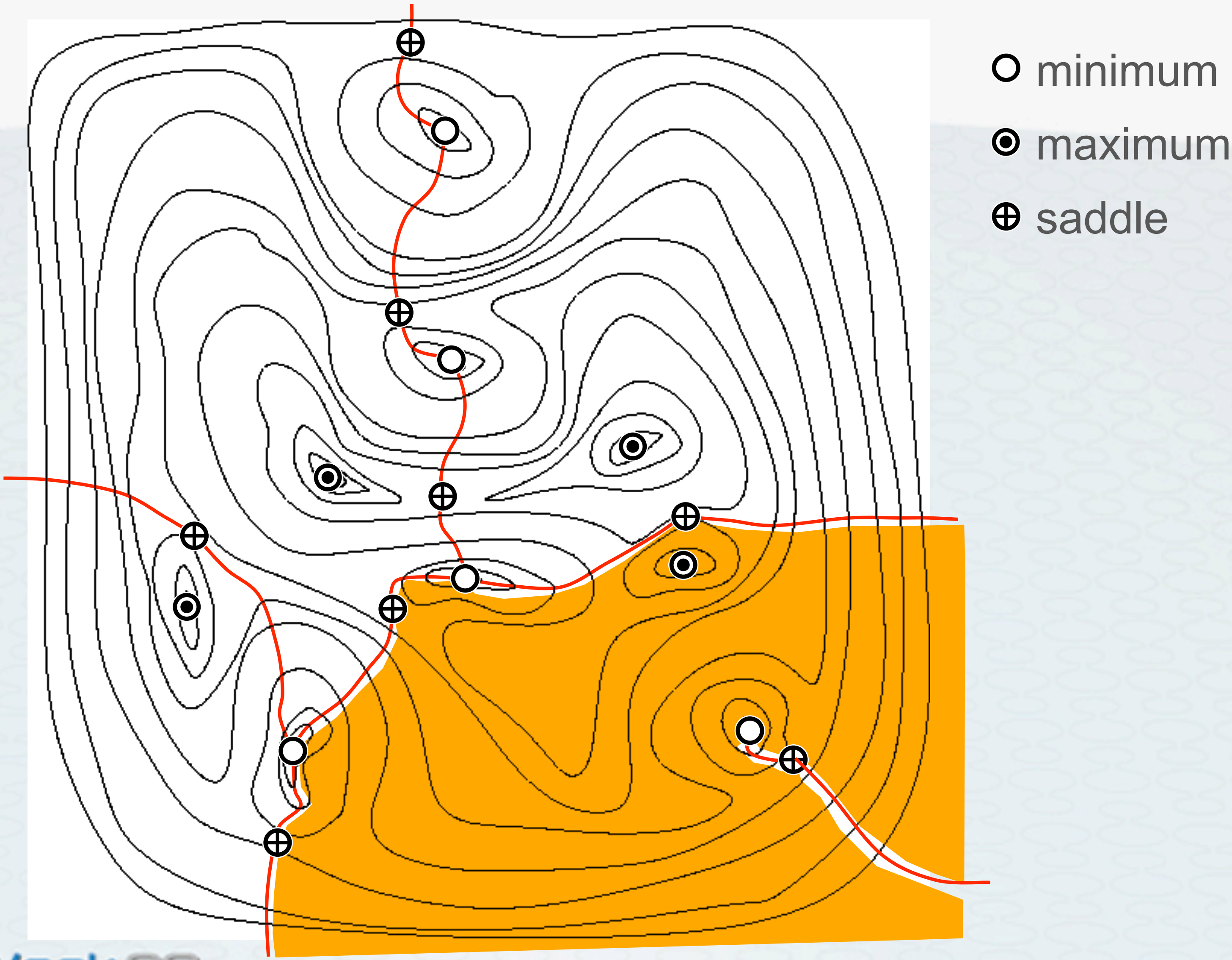


- minimum
- maximum
- saddle

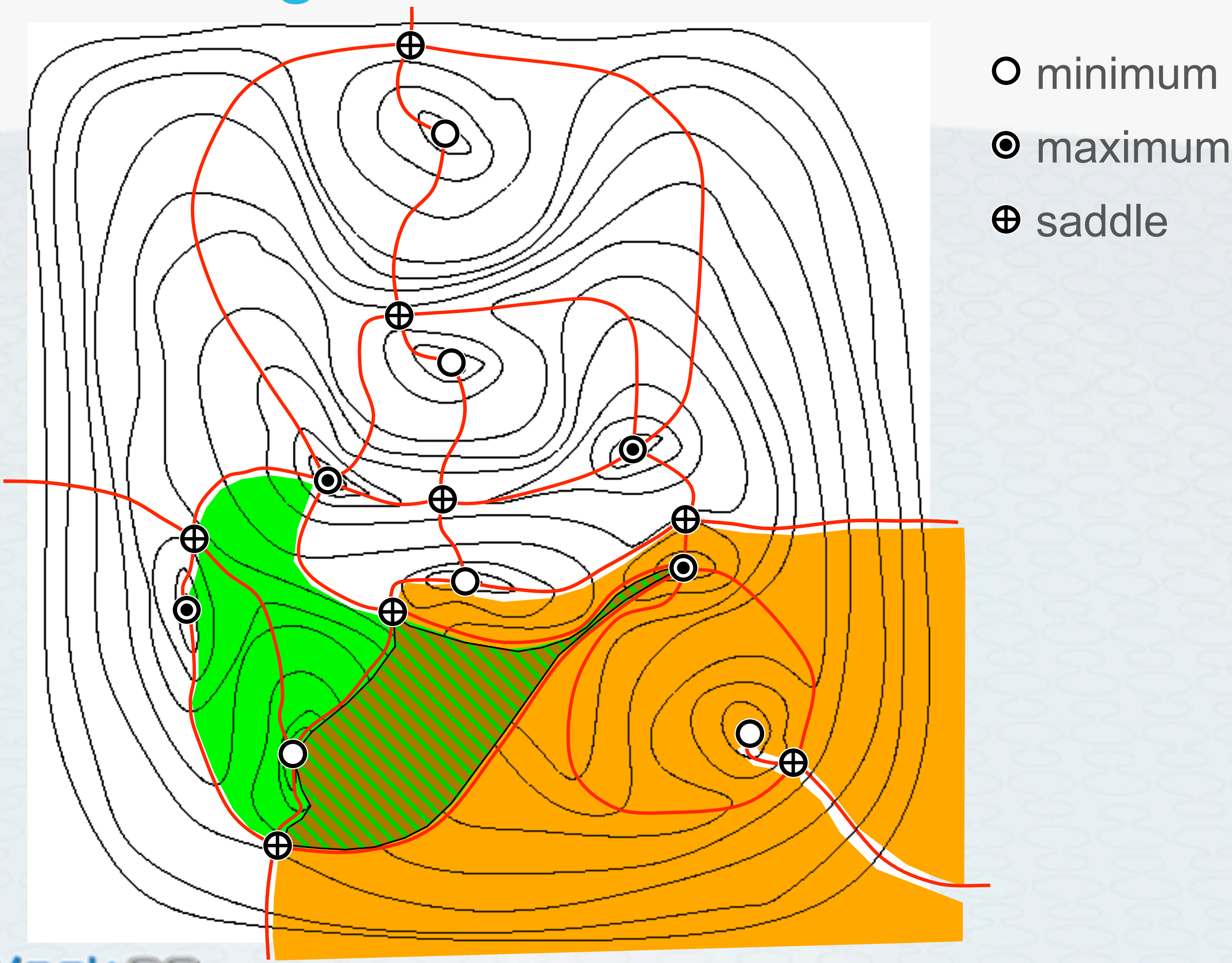
Lines of Steepest Ascent



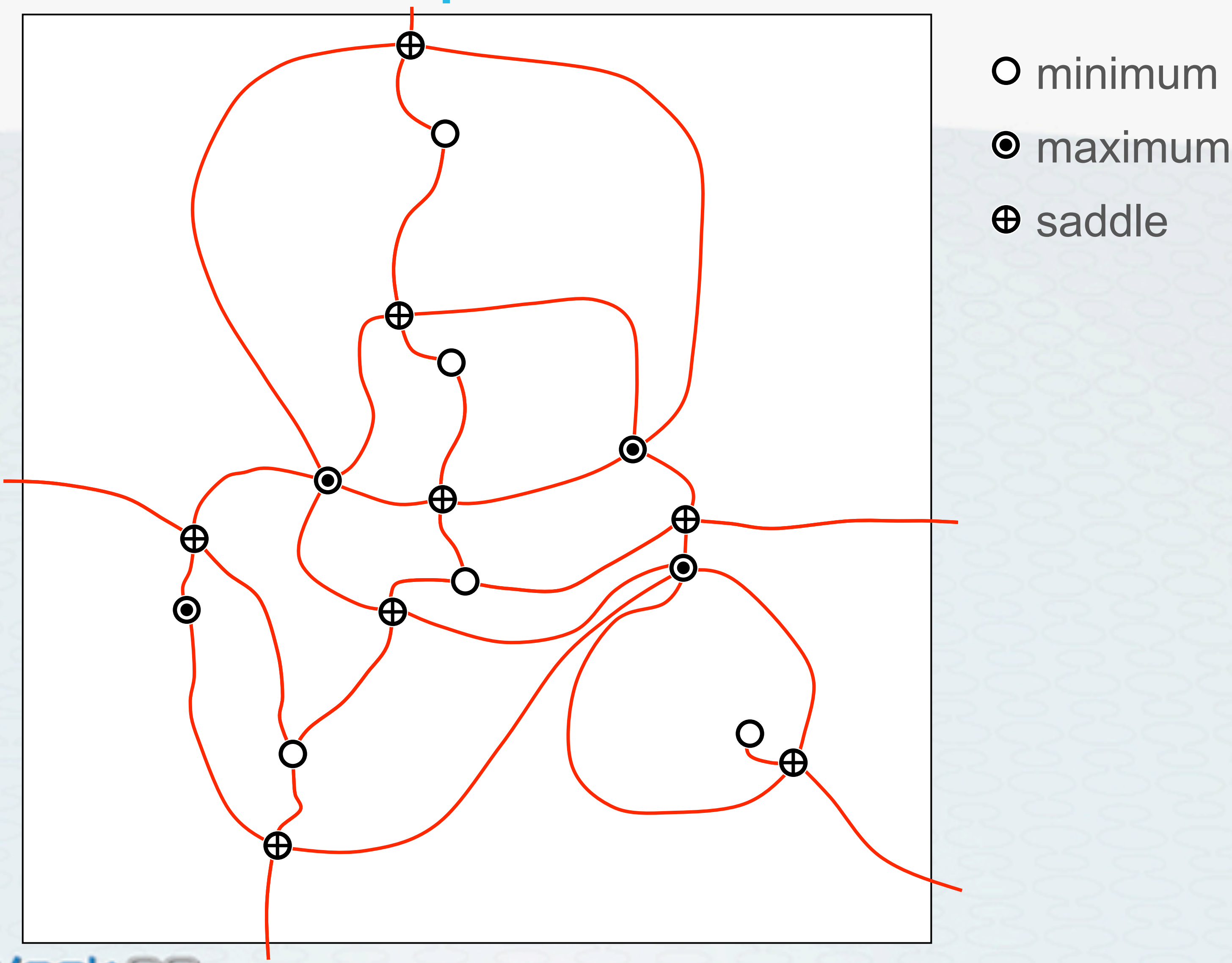
Complex of Unstable Manifolds



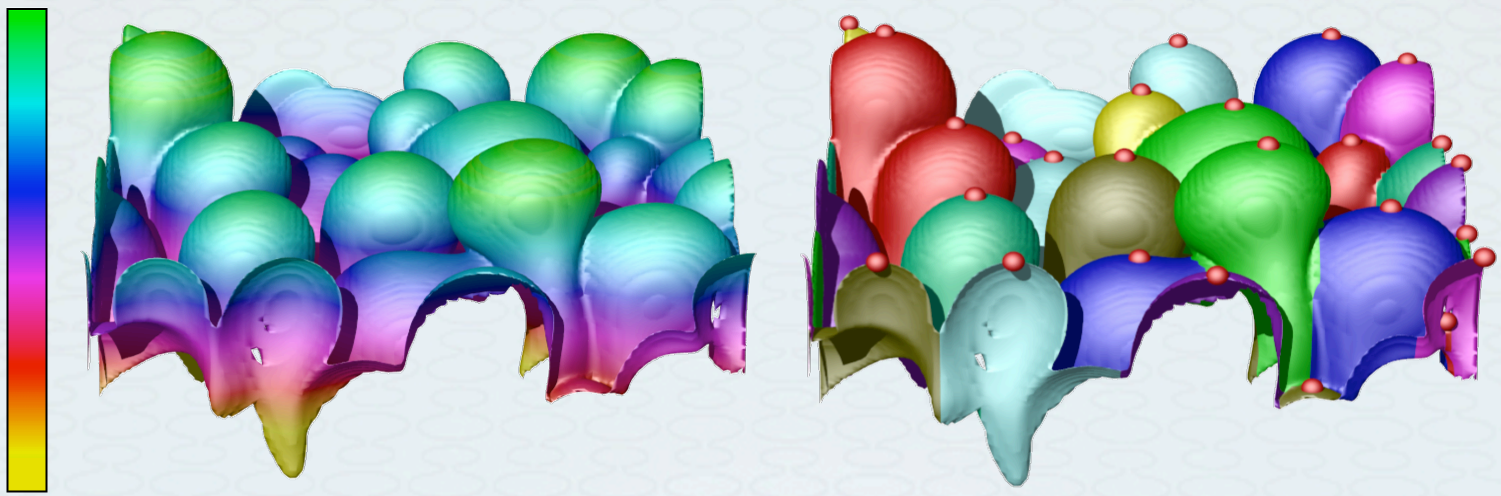
Four-sided Regions



Morse-Smale Complex



Gradient-line-based Segmentation



Further Relevant Reading/ Topics not Covered Here

- Jacobi Sets for
  - time-varying data
  - comparison of scalar functions
- Contour Spectrum

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  - Edelsbrunner's upcoming book on computational topology
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  - G.H. Weber, G. Scheuermann, H. Hagen, B. Hamann, Exploring Scalar Fields Using Critical Isovalues, Proc. IEEE Visualization 2002, pp. 171-178, 2002.

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- Morse-Smale Complex
  - Edelsbrunner references listed for critical points

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